

ANALYSIS SYLLABUS

Metric Space Topology

Metrics on R^n , compactness, Heine-Borel Theorem, Bolzano-Weierstrass Theorem.

Sequences and Series

Limits and convergence criteria.

Functions defined on R^n

Continuity, uniform continuity, uniform convergence, Weierstrass Comparison Test, uniform convergence and limits of integrals, Ascoli's Theorem.

Differentiability

Differentiable functions, chain rule, local maxima and minima.

Transformations on R^n

Derivative as a linear transformation, inverse function theorem, implicit function theorem.

Riemann integration on R^n

Riemann-integrable functions, improper integrals; line integrals, surface integrals; change of variable formula; Green's theorem, Stokes' theorem, Gauss' divergence theorem.

References

Bartle, R. G, and Sherbert, D. R., *Introduction to Real Analysis*. John Wiley & Sons (1992)

R. Creighton Buck, *Advanced Calculus*. McGraw-Hill (1978)

Walter Rudin, *Principles of Mathematical Analysis*. McGraw-Hill (1976)

Strichartz, R. S., *The Way of Analysis*. Jones and Bartlett (1995)

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Analysis Qualifying Exam

August, 1996

You should attempt all nine of the following problems. Good luck!

1. Let X be the metric space

$$X = \{(x, y) \in \mathbb{R}^2 : y \geq |x|^{2/3}\}$$

with the usual Euclidean distance, and define $f : X \rightarrow \mathbb{R}$ by $f(x, y) = \frac{xy^3}{x^4 + y^4}$ for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Decide whether or not f is continuous at $(0, 0)$, and prove your answer by applying the $\epsilon - \delta$ definition of continuity. Is f continuous at $(0, 0)$ when considered as a mapping from \mathbb{R}^2 into \mathbb{R} ? Prove your answer.

2. Define $g : [-1, 1] \rightarrow \mathbb{R}$ by $g(x) = (-1)^k/k^2$ for $|x| \in (1/(k+1), 1/k]$, $k = 1, 2, \dots$, and $g(0) = 0$. Decide whether or not g is differentiable at 0, and prove your answer.

3. Let $\{a_n\}_{n=0}^{\infty}$ be the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \dots\}$. (Thus $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$.) Show that the series $\sum_{n=0}^{\infty} \frac{1}{a_n}$ converges.

4. Compute $\int_{\Phi} \text{curl } F \cdot N dA$, where F is the vector field $F(x, y, z) = \frac{(-z, y, x)}{\sqrt{x^2 + z^2 + 1}}$, $\Phi : [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ is the surface $\Phi(r, \theta) = (r \cos \theta, r^2, r \sin \theta)$, N is a unit normal vector on Φ , and dA is the surface area element.

5. Let E be an open set in \mathbb{R}^n , and let $F : E \rightarrow \mathbb{R}^n$ be C^1 . Show that, if the function $|F|^2$ has a nonzero relative minimum at a point $x_0 \in E$, then the linear transformation $F'(x_0)$ must be singular.

6. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be continuous, and assume that $\lim_{x \rightarrow \infty} f(x)$ exists and is a finite number L . What can be said about

$$\lim_{n \rightarrow \infty} \int_0^1 f(nx) dx ?$$

Prove your answer.

7. Let A be the set of real numbers in $[0, 1]$ whose decimal expansions contain only the digits 3 and 8. Is A countable? Is A dense in $[0, 1]$? Is A closed? Prove your answers.

8. Let $E \subset \mathbb{R}^2$ be open and nonempty. Prove that there is no one-to-one, C^1 function mapping E into \mathbb{R} .

9. Let $E \subset \mathbb{R}^2$ be open, and let $F : E \rightarrow \mathbb{R}$ have continuous second order derivatives in E . Denote by f'' the matrix of second partial derivatives $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.

- a. Show that the set of points in E at which f'' has repeated eigenvalues is closed relative to E .
 b. Suppose that f'' is positive definite in E ; that is, suppose that, for each $x \in E$ and $h \in \mathbb{R}^2 - \{0\}$, $(f''(x)h) \cdot h > 0$. Show that, for any compact subset $K \subset E$, there is a positive constant ϵ such that

$$(f''(x)h) \cdot h \geq \epsilon|h|^2$$

for all $x \in K$ and all $h \in \mathbb{R}^2$.

Tier 1 Analysis Examination

August 1997

1. Does $a_k = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdots (2k+1)}{2 \cdot 4 \cdot 6 \cdots 2k}$ converge or diverge? Prove your assertion.

2. Let $S \subset \mathbb{R}^3$ be the "tin can without a lid".

$$\{(x, y, z) : x^2 + y^2 = 1, 0 \leq z \leq 1\} \cup \{(x, y, z) : x^2 + y^2 \leq 1, z = 0\}$$

Compute the flow $\iint_S F \cdot N \, dA$ "out" of the can if $F = (x(y+z), -zy, -zy)$

3. Definition: A transformation of class C^1 $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is called volume preserving if for every cube $C \subset \mathbb{R}^3$, with faces parallel to the coordinate planes, $\text{volume}(F(C)) = \text{volume}(C)$.

(i) Show that $F(x, y, z) = (x + y, z - 4, z^2 - y)$ is volume preserving.

(ii) Show that if $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is volume preserving then the determinant of its derivative G' equals ± 1 , and G maps open sets into open sets.

4. Let $f_n(x) = \int_{1/2}^x \arctan^2(t/n) \, dt \quad n = 1, 2, \dots$

(i) Show that $\sum_{n=1}^{\infty} f'_n(x)$ (sum of derivatives) is uniformly convergent on $[-1, 1]$.

(ii) Show that $g(x) = \sum_{n=1}^{\infty} f_n(x)$ is differentiable for all x .

5. Let I_j be a countable family of closed intervals whose interiors are pairwise disjoint and such that $\bigcup I_j = [0, 1]$. Show directly (without fancy integration theorems) that $\sum_{j=1}^{\infty} |I_j| = 1$.

6. Give a counterexample to this statement: Every $f : \mathbb{R} \rightarrow \mathbb{R}$ with the property that $f^{-1}(K)$ is compact for any compact K is continuous.

7. Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^∞ function with $0 \in \mathbb{R}^2$ a critical point. Suppose the matrix of second partials $\left(\frac{\partial^2 g}{\partial x_i \partial x_j}(0)\right)$ has eigenvalues -2 and 0 . Show that the origin is NOT a local minimum of g .

8. Definition: A metric space is said to have property Z if every sequence with exactly one cluster point converges. (Recall that every neighborhood of a cluster point of $\{x_n\}$ contains infinitely many x_n).

(i) Give an example of a metric space that has property Z and an example of a metric space that does not.

(ii) What properties of metric spaces are implied by or equivalent to property Z ?

ANALYSIS EXAMINATION

January, 1998

Instructions: Answer all seven questions. Each of the seven questions is equally weighted.
 Notation: \mathbf{R} denotes the set of all real numbers.

1. State whether each of the following limit exists, and prove your assertions.

$$(a) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^3}{x^4 + y^4} \quad (b) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^3}{x^4 + y^6}$$

2. Suppose $f : \mathbf{R} \rightarrow \mathbf{R}$ is a uniformly continuous function on \mathbf{R} . If

$$f_k(x) = k \int_x^{x+(1/k)} f(t) dt \quad \text{for } x \in \mathbf{R} \text{ and } k = 1, 2, 3, \dots,$$

prove that the sequence $\{f_k\}$ converges to f uniformly on \mathbf{R} .

3. Compute the surface integral $\iint_S (x^2 + y^2) dA$, where S is the boundary of the set $\{(x, y, z) \in \mathbf{R}^3 : \sqrt{x^2 + y^2} \leq z \leq 1\}$, and dA denotes the surface area element.

4. Let $\pi_1 : \mathbf{R}^2 \rightarrow \mathbf{R}$ and $\pi_2 : \mathbf{R}^2 \rightarrow \mathbf{R}$ be the projection maps

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y, \quad \text{for } (x, y) \in \mathbf{R}^2,$$

and let S be the horizontal strip $S = \{(x, y) \in \mathbf{R}^2 : -1 \leq y \leq 1\}$. State whether each of the following assertions is TRUE or FALSE, and prove your assertions.

- (a) If E is a closed subset of \mathbf{R}^2 such that $E \subset S$, then the image $\pi_1(E)$ must be a closed subset of \mathbf{R} .
 (b) If E is a closed subset of \mathbf{R}^2 such that $E \subset S$, then the image $\pi_2(E)$ must be a closed subset of \mathbf{R} .
5. If f is a continuous function on $[0, 1]$, prove that

$$\lim_{t \uparrow 1} \left[(1-t) \sum_{k=0}^{\infty} t^k f(t^k) \right] = \int_0^1 f(x) dx.$$

6. Let Ω be a bounded, connected open set in \mathbf{R}^n , and let f be a continuous real-valued function on the closure of Ω . Suppose that f is of class C^∞ on the set Ω , and suppose that for each point $p \in \Omega$ there is at least one index $i \in \{1, 2, \dots, n\}$ such that

$$\frac{\partial^2 f}{\partial x_i^2}(p) < 0.$$

If

$$f(p) \geq 0 \quad \text{for every point } p \text{ in the boundary of } \Omega,$$

prove that

$$f(p) \geq 0 \quad \text{for every point } p \text{ in } \Omega.$$

7. Let Ω be a convex open set in \mathbf{R}^2 . Let $f : \Omega \rightarrow \mathbf{R}$ and $g : \Omega \rightarrow \mathbf{R}$ be functions of class C^∞ , and assume that for each point $p \in \Omega$ we have

$$\frac{\partial f}{\partial x}(p) \geq 5, \quad \frac{\partial g}{\partial y}(p) \geq 5, \quad \left| \frac{\partial f}{\partial y}(p) \right| \leq 1, \quad \left| \frac{\partial g}{\partial x}(p) \right| \leq 1.$$

Define $T : \Omega \rightarrow \mathbf{R}^2$ by $T(p) = (f(p), g(p))$ for each point $p \in \Omega$. Prove that the image $T(\Omega)$ is an open subset of \mathbf{R}^2 , and that T is a one-to-one mapping from Ω onto $T(\Omega)$.

Tier 1 Analysis Examination

August 1998

1. Consider the sequence of functions $f_k(x) := \{\sin(kx)\}$, $k = 1, 2, \dots$, and observe that $\sin(kx) = 0$ if $x = m\pi/k$ for all integers m . Given an arbitrary interval $[a, b]$, show that $\{f_k\}$ has no subsequence that converges uniformly on $[a, b]$.

2.

- (a) Given a sequence of functions f_k defined on $[0, 1]$, define what it means for $\{f_k\}$ to be equicontinuous.
- (b) Let $G(x, y)$ be a continuous function on \mathbf{R}^2 and suppose for each positive integer k , that g_k is a continuous function defined on $[0, 1]$ with the property that $|g_k(y)| \leq 1$ for all $y \in [0, 1]$. Now define

$$f_k(x) := \int_0^1 g_k(y)G(x, y) dy.$$

Prove that the sequence $\{f_k\}$ is equicontinuous on $[0, 1]$.

3. Let $\Omega \subset \mathbf{R}^n$ be an open connected set and let $\Omega \xrightarrow{f} \Omega$ be a C^1 transformation with the property that determinant of its Jacobian matrix, $|Jf|$, never vanishes. That is, $|Jf(x)| \neq 0$ for each $x \in \Omega$. Assume also that $f^{-1}(K)$ is compact whenever $K \subset \Omega$ is a compact set. Prove that $f(\Omega) = \Omega$.
4. Let $G(x, y)$ be a continuous function defined on \mathbf{R}^2 . Consider the function f defined for each $t > 0$ by

$$f(t) := \int \int_{x^2+y^2 < t^2} \frac{G(x, y)}{\sqrt{t^2 - x^2 - y^2}} dx dy.$$

Prove that

$$\lim_{t \rightarrow 0^+} f(t) = 0.$$

5. Let (X, \mathbf{d}) be a compact metric space and let \mathcal{G} be an arbitrary family of open sets in X . Prove that there is a number $\lambda > 0$ with the property that if $x, y \in X$ are points with $\mathbf{d}(x, y) < \lambda$, then there exists an open set $U \in \mathcal{G}$ such that both x and y belong to U .
6. Let $\Gamma := \{(x, y, z) \in \mathbf{R}^3 : e^{xy} = x, x^2 + y^2 + z^2 = 10\}$. The Implicit Function theorem ensures that Γ is a curve in some neighborhood of the point $p = (e, \frac{1}{e}, \sqrt{10 - e^2 - \frac{1}{e^2}})$. That is, there is open interval $I \subset \mathbf{R}^1$ and a C^1 mapping $I \xrightarrow{\gamma} \Gamma$ such that $\gamma(0) = p$. Find a unit vector v such that $v = \pm \frac{\gamma'(0)}{|\gamma'(0)|}$.
7. Suppose that a hill is described as $\{(x, y, z) \in \mathbf{R}^3 : (x, y, f(x, y))\}$ where $f(x, y) = x^3 + x - 4xy - 2y^2$. Suppose that a climber is located at $p = (1, 2, -14)$ on the hill and wants to move from p to another location on the hill without changing elevation. In which direction should the climber proceed from p ? Express your answer in terms of a vector and completely justify your answer.

8. Suppose g and f_k ($k = 1, 2, \dots$) are defined on $(0, \infty)$, are Riemann integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_k| \leq g$, $f_k \rightarrow f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x) dx < \infty.$$

Prove that

$$\lim_{k \rightarrow \infty} \int_0^\infty f_k(x) dx = \int_0^\infty f(x) dx.$$

Tier 1 Analysis Examination

January 1999

1. Prove that the function

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

satisfies $f'(0) > 0$, but that there is no open interval containing 0 on which f is increasing.

2. Let $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a mapping defined by $F(x, y) = (u, v)$ where

$$\begin{aligned} u &= u(x, y) = x \cos(y) \\ v &= v(x, y) = y \cos(x). \end{aligned}$$

Note that $F(-\pi/3, \pi/3) = (-\pi/6, \pi/6)$.

- (i) Show that there exist neighborhoods U of $(-\pi/3, \pi/3)$, V of $(-\pi/6, \pi/6)$, and a differentiable function $G: V \rightarrow U$ such that F restricted to U is one-to-one, $F(U) = V$ and $G(F(x, y)) = (x, y)$ for every $(x, y) \in U$.
- (ii) Let U, V and G be as in part (i), and write

$$G(u, v) = (x, y), \text{ with } x = x(u, v), y = y(u, v).$$

Find

$$\frac{\partial x}{\partial u}(-\pi/6, \pi/6) \quad \text{and} \quad \frac{\partial y}{\partial v}(-\pi/6, \pi/6).$$

3. Beginning with $a_1 \geq 2$, define a sequence recursively by $a_{n+1} = \sqrt{2 + a_n}$. Show that the sequence is monotone and compute its limit.
4. Let $f: K \rightarrow \mathbf{R}^n$ be a one-to-one continuous mapping, where $K \subset \mathbf{R}^n$ is a compact set. Thus, the mapping f^{-1} is defined on $f(K)$. Prove that f^{-1} is continuous.
5. Let S denote the 2-dimensional surface in \mathbf{R}^3 defined by $F: D \rightarrow \mathbf{R}^3$ where $D = \{(x, y) : x^2 + y^2 \leq 4\}$ and $F(x, y) = (x, y, 6 - (x^2 + y^2))$. Let ω be the differential 1-form in \mathbf{R}^3 defined by $\omega = yz^2 dx + xz dy + x^2 y^2 dz$. After choosing an orientation of S , evaluate the integral

$$\int_S z dx \wedge dy + d\omega.$$

6. Let $f: U \rightarrow \mathbf{R}^1$ where $U := (0, 1) \times (0, 1)$. Thus, $f = f(x, y)$ is a function of two variables. Assume for each fixed $x \in (0, 1)$, that $f(x, \cdot)$ is a continuous function of y . Let \mathcal{F} denote the countable family of functions $f(\cdot, r)$ where $r \in (0, 1)$ is a rational number. Thus, for each rational number $r \in (0, 1)$, $f(\cdot, r)$ is a function of x . Assume that the family \mathcal{F} is equicontinuous. Now prove that f is a continuous function of x and y ; that is, prove that $f: U \rightarrow \mathbf{R}^1$ is a continuous function.

7. Let $f_1 \geq f_2 \geq f_3 \geq \dots$ be a sequence of real-valued continuous functions defined on the closed unit ball $B \subset \mathbf{R}^n$ such that $\lim_{k \rightarrow \infty} f_k(x) = 0$ for each $x \in B$. Prove that $f_k \rightarrow 0$ uniformly on B . This is a special case of Dini's theorem. You may not appeal to Dini's theorem to answer the problem.
8. Let $f: \mathbf{R}^1 \rightarrow \mathbf{R}^1$ be a nonnegative function satisfying the Lipschitz condition $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$ for all $x_1, x_2 \in \mathbf{R}^1$ and where $K > 0$. Suppose that

$$\int_0^\infty f(x) dx < \infty.$$

Prove that

$$\lim_{x \rightarrow \infty} f(x) = 0.$$

9. Let F be a nonnegative, continuous real-valued function defined on the infinite strip $\{(x, y) : 0 \leq x \leq 1, y \in \mathbf{R}^1\}$ with the property that $F(x, y) \leq 4$ for all $(x, y) \in [0, 1] \times [0, 2]$. Let f_n be a continuous piecewise-linear function from $[0, 1]$ to \mathbf{R}^1 such that $f_n(0) = 0$, f_n is linear on each interval of the form $[\frac{i}{n}, \frac{i+1}{n}]$, $i = 0, 1, \dots, n-1$, and for $x \in (\frac{i}{n}, \frac{i+1}{n})$, $f'_n(x) = F(\frac{i}{n}, f_n(\frac{i}{n}))$. Prove that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that f_{n_k} converges uniformly to a function f on $[0, 1/2]$.

TIER I ANALYSIS EXAMINATION

August 1999

There is no unusual notation in this exam: \mathbf{R} stands for the real line, \mathbf{R}^n for n -dimensional Euclidean space, and $\|x\|$ for the Euclidean norm of a vector $x \in \mathbf{R}^n$ (distance from x to 0). You must do eight of the following problems. Please indicate which of the nine problems should not be graded.

1. Let $f : (0, \infty) \rightarrow \mathbf{R}$ be a bounded continuous function. Show that there exists $c \in (0, \infty)$ such that $\int_0^\infty e^{-x} f(x) dx = f(c)$.
2. Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a continuous function such that $\|f(x)\| < \|x\|$ for every point $x \neq 0$. Fix a point $x_1 \in \mathbf{R}^n$, and define recursively $x_{n+1} = f(x_n)$ for $n \geq 1$. Show that the sequence $(x_n)_{n=1}^\infty$ converges to 0.
3. Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a mapping of class C^1 such that the Jacobian determinant $J_f(x)$ is different from zero for all points x . Assume in addition that $\{x : \|f(x)\| < M\}$ is a bounded set for every $M > 0$. Show that f is onto. That is, show that for every $y \in \mathbf{R}^n$ there exists at least one point $x \in \mathbf{R}^n$ such that $f(x) = y$.
4. Consider the surface S in \mathbf{R}^3 consisting of all points of coordinates (x, y, z) such that $x^2 + y^2 + z^2 = 1$ and $x \geq \frac{1}{2}$, and choose an orientation for S . Calculate the integral $\int_S \omega$, where the 2-form ω is defined by

$$\omega(x, y, z) = x dx \wedge dy + y dy \wedge dz + z dz \wedge dx$$

for $(x, y, z) \in \mathbf{R}^3$.

5. Denote by $D = \{(x, y) : x > 0\}$ the right half-plane in \mathbf{R}^2 , and let f be a function of class C^1 defined on D . Assume that

$$\frac{\partial f}{\partial x}(x, y) \leq \frac{1}{\sqrt{x}} \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) \leq 1$$

for all $(x, y) \in D$. Show that f is uniformly continuous on D .

6. Assume that the function $f : \mathbf{R} \rightarrow \mathbf{R}$ is differentiable at every point, and $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are two sequences converging to zero with $a_n < b_n$ for all n . Do the quotients

$$\frac{f(b_n) - f(a_n)}{b_n - a_n}$$

necessarily converge to $f'(0)$? (Prove if yes, give a counterexample if no.)

7. Let $f : [0, 1] \times [0, 1] \rightarrow \mathbf{R}$ be a continuous function, and define $g : [0, 1] \rightarrow \mathbf{R}$ by $g(x) = \max_{y \in [0, 1]} f(x, y)$. Show that g is continuous.
8. Let $(a_n)_{n=1}^\infty$ be a sequence of real numbers, and define $s_n = \sum_{k=1}^n a_k$. Assume that $\lim_{n \rightarrow \infty} \sqrt{n} a_n = 1$ and prove that $\lim_{n \rightarrow \infty} s_n / \sqrt{n} = 2$.
9. Consider a complete metric space (X, d) , and a sequence $F_1 \supseteq F_2 \supseteq \dots$ of nonempty, closed subsets of X . Assume that for each n , the set F_n can be covered by a finite number of balls of radius $1/n$. For each n , select a point $x_n \in F_n$. Prove that the sequence $(x_n)_{n=1}^\infty$ has a convergent subsequence.

Tier 1 Analysis Exam
January 2000

1. Let Ω be an open set in \mathbb{R}^2 . Let u be a real-valued function on Ω . Suppose that for each point $a \in \Omega$ the partial derivatives $u_x(a)$ and $u_y(a)$ exist and are equal to zero.
 - (i) Prove that u is locally constant, *i.e.* for every point in Ω there is a neighborhood on which u is a constant function.
 - (ii) Prove that if Ω is connected, then u is a constant function on Ω .
2. Let S be the surface in the Euclidean space \mathbb{R}^3 given by the equation $x^2 + y^2 - z^2 = 1$, $0 \leq z \leq 1$, oriented so that the normal vector points away from the z -axis. Find $\int_S \mathbf{F} \cdot d\mathbf{S}$, where \mathbf{F} is the vector field defined by

$$\mathbf{F}(x, y, z) = (-xy^2 + z^5, -x^2y, (x^2 + y^2)z) .$$

3. Let $f(x) = e^x - \cos x$ for $x \in \mathbb{R}$.
 - (i) Show that on a neighborhood around $x = 0$, f has an inverse function g with $g(0) = 0$.
 - (ii) Compute $g''(0)$.
 - (iii) Show that there exists $a > 0$ such that $f : (-a, \infty) \rightarrow (f(-a), \infty)$ is a homeomorphism.
4. For positive numbers k_1, k_2, k_3, \dots we define $[k_1] = \frac{1}{k_1}$, $[k_1, k_2] = \frac{1}{k_1 + [k_2]}$, $[k_1, k_2, k_3] = \frac{1}{k_1 + [k_2, k_3]}$, and inductively, $[k_1, \dots, k_{n+1}] = \frac{1}{k_1 + [k_2, \dots, k_{n+1}]}$. Prove that $\lim_{n \rightarrow \infty} [k_1, \dots, k_n]$ exists if $k_n \geq 2$ for all n .
5. Two circular holes of radius 1 *in* are drilled from the centers of two faces of a solid cube of volume 64 in^3 . Compute the volume of the remaining solid.
6. Let $\varphi_1, \varphi_2, \varphi_3, \dots$ be non-negative continuous functions on $[-1, 1]$ such that
 - (i) $\int_{-1}^1 \varphi_k(t) dt = 1$ for $k = 1, 2, 3, \dots$;
 - (ii) for every $\delta \in (0, 1)$ $\lim_{k \rightarrow \infty} \varphi_k = 0$ uniformly on $[-1, -\delta] \cup [\delta, 1]$.

Prove that for every continuous function $f : [-1, 1] \rightarrow \mathbb{R}$ we have

$$\lim_{k \rightarrow \infty} \int_{-1}^1 f(t) \varphi_k(t) dt = f(0) .$$

7. Suppose $\lim_{n \rightarrow \infty} a_n = a$, $\lim_{n \rightarrow \infty} b_n = b$, and let

$$c_n = \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} .$$

Prove that $\lim_{n \rightarrow \infty} c_n = ab$.

8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a uniformly continuous function on \mathbb{R} . Prove that there exist positive constants A and B such that

$$|f(x)| \leq A|x| + B \quad \text{for all } x \in \mathbb{R} .$$

9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. Suppose $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 1$. Prove that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} x_n = \infty$ and $\lim_{n \rightarrow \infty} f'(x_n) = 1$.

Tier I Analysis Exam, Fall 2000

It is important to justify your answers. A correct answer, without justification (for example to #3 or #4) will receive no credit.

1. Evaluate the limit

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n} \right\}$$

by interpreting it as a definite integral.

2. Consider the 1-form F defined on $\mathbb{R}^2 \setminus \{0\}$ by

$$F = \frac{xdy - ydx}{x^2 + y^2}$$

- (a) Evaluate $\int_{\partial C} F$, where C is the unit square, $[-1, 1] \times [-1, 1]$, in \mathbb{R}^2 , positively oriented.
- (b) Is F exact on $\mathbb{R}^2 \setminus \{0\}$?
3. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous, real-valued functions on $[0, 1]$ that converges uniformly to a function f on $[0, 1]$. Must f have a zero in $[0, 1]$ (i.e. $f(x) = 0$ for some $x \in [0, 1]$) if each f_n has a zero in $[0, 1]$?
4. Does the series $\sum_{n=1}^{\infty} \frac{\cos(\log n)}{n}$ converge or diverge?

5. Let f be a continuous function on $[0, 1]$. Show that

$$\int_0^1 f(x) \sin(nx) dx \rightarrow 0$$

as $n \rightarrow \infty$.

6. Is the function $f(x) = \sqrt{x}$ uniformly continuous on $[0, \infty)$?

7. Consider the function $f = \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Show that $\frac{\partial f}{\partial x}$ exists everywhere on \mathbb{R}^2 but that $\frac{\partial f}{\partial x}$ is not continuous everywhere.

8. Let

$$X = \left\{ \begin{array}{l} f : [0, 2\pi] \rightarrow \mathbb{R} : f \text{ is continuous} \\ \text{and } |f(x)| \leq 1 \text{ for all } x \in [0, 2\pi] \end{array} \right\}$$

Put a metric d on X by defining

$$d(f, g) = \sqrt{\int_0^{2\pi} (f(x) - g(x))^2 dx}$$

(You may assume that d actually does define a metric on X .)

Is (X, d) compact?

9. Let $\mathbb{R}^{2 \times 2}$ denote the set of all real 2×2 matrices. Make it a metric space by identifying $\mathbb{R}^{2 \times 2}$ with the 4-dimensional Euclidean space \mathbb{R}^4 via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \approx (a, b, c, d)$$

Let $X \subset \mathbb{R}^{2 \times 2}$ denote the subset of all invertible 2×2 matrices. Is X connected?

10. Consider the two equations

$$F_1(x, y, u, v) \equiv e^{x^2 - y^2} u^5 - v^3 = 0$$

$$F_2(x, y, u, v) \equiv e^{u^2 - v^2} x^2 - y^2 = 0$$

Prove that there exists a neighborhood, U , of $(1, 1) \in \mathbb{R}^2$ and functions $u(x, y)$ and $v(x, y)$ on U with $u(1, 1) = v(1, 1) = 1$ such that

$$F_1(x, y, u(x, y), v(x, y)) = F_2(x, y, u(x, y), v(x, y)) = 0 \quad \forall x, y \in U.$$

Find $\left. \frac{\partial u(x, y)}{\partial x} \right|_{(1, 1)}$.

11. Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ ($n = 1, 2, \dots$) be an equicontinuous sequence of functions. If $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for each $x \in \mathbb{R}$, does it follow that the convergence is uniform on \mathbb{R} ?

Tier 1 Exam
January, 2001

1. Compute

$$\int_S \operatorname{curl} F \cdot N \, dA,$$

where S is that part of the surface $y = x^2 + z^2$ in \mathbb{R}^3 for which $0 \leq y \leq 1$, N is the unit normal to S pointing toward the y -axis, dA is the area element, and $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the mapping

$$F(x, y, z) = e^{x^2+z^2}(z, y, -x).$$

2. Suppose that $f: [0, \infty) \rightarrow \mathbb{R}^1$ is a nonnegative, uniformly continuous function and that

$$\int_0^\infty f(x) \, dx < \infty.$$

Prove that $\lim_{x \rightarrow \infty} f(x) = 0$.

3. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^1$ be continuous, and define

$$g(y) = \int_0^1 f(x, y) \, dx.$$

Assume that $\frac{\partial f}{\partial y}$ is continuous on \mathbb{R}^2 , and compute $g'(y)$. Prove your result.

4. The function $\frac{1}{32}x^4 + x^2y^2 - x^3 - y^3 - xy^3$ has critical points at $(24, 0)$ and $(0, 0)$. By a careful analysis, determine whether each point is a local maximum, local minimum or a point which is neither a local maximum nor a local minimum.
5. Let $f: B \rightarrow \mathbb{R}^1$ be a uniformly continuous function, where $B \subset \mathbb{R}^n$ is an open ball. Prove that there is a uniformly continuous function F defined on the closure of B such that F restricted to B is equal to f .

6. Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined for points $x = (x, y)$ by

$$g(x, y) = (x^2 + y^2 - |x^2 - y^2|, x^2 + y^2 + |x^2 - y^2|).$$

- Give the definition of the differential of g at x_0 , denoted by $dg(x_0)$.
- Determine those points $x_0 \in \mathbb{R}^2$ where $dg(x_0)$ exists and where it does not exist. In both cases, justify your answer. Be sure to analyze the case $x_0 = 0$.
- Find those points $x_0 \in \mathbb{R}^2$ where g locally has a differentiable inverse and where it does not. In both cases, justify your answer.

7. Let n be an integer greater than 1, and consider the following statement: If ω is a differential 2-form on \mathbb{R}^n with the property that $\omega \wedge \lambda = 0$ for every differential 1-form λ , then ω must be the zero form. For what n is the above statement true? For what n is it false? Prove your answers.

8. (a) Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by

$$F(x, y, z) = (z^2 + xy - 1, z^2 + x^2 - y^2 - 2)$$

and observe that $F(\mathbf{a}) = (0, 0)$ where $\mathbf{a} = (-1, 0, 1)$. Prove that there exist an open interval (a, b) , a C^1 curve of the form $\gamma(t) = (f(t), g(t), h(t))$ with $a < t < b$, and an open set $U \subset \mathbb{R}^3$ containing \mathbf{a} such that

$$U \cap F^{-1}(0, 0) = \{\gamma(t) : a < t < b\}$$

(b) Compute $\gamma'(t_0) = (f'(t_0), g'(t_0), h'(t_0))$ where $\gamma(t_0) = \mathbf{a}$.

9. Let $f: [0, 1] \rightarrow \mathbb{R}^1$ be defined by

$$f(x) = \begin{cases} \frac{1}{2^k} & \text{if } \frac{1}{k+1} < x \leq \frac{1}{k}, k = 1, 2, \dots \\ 0 & \text{for } x = 0. \end{cases}$$

(a) For any given $\varepsilon > 0$, show how to construct a partition P of the interval $[0, 1]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

($U(P, f)$ and $L(P, f)$ are the upper and lower Riemann sums for f over the partition P).

(b) Find an expression for

$$\int_0^1 f(x) dx.$$

and justify your answer.

TIER I ANALYSIS EXAMINATION

August 24, 2001

NOTATION: For $x \in \mathbb{R}^n$, let $|x|$ denote the Euclidean norm of x (i.e. the Euclidean distance of x from the origin $\vec{0} \in \mathbb{R}^n$).

1. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that

$$\frac{\partial f}{\partial x_1}(\vec{0}) = \frac{\partial f}{\partial x_2}(\vec{0}) = 0.$$

- (a) Does it follow that f differentiable at $x = \vec{0}$? Explain.
 (b) Prove or give a counterexample that f is continuous at $x = \vec{0}$.

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued, differentiable functions on the real line such that

- (a) $f_n(x) \rightarrow 0$ for each $x \in [0, 1]$
 (b) $|f'_n(x)| \leq 1$ for all $x \in [0, 1]$ and all $n = 1, 2, \dots$

Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[0, 1]$ to 0.

3. Let $E \subset \mathbb{R}^n$ be nonempty. For $x \in \mathbb{R}^n$ define

$$D(x) \equiv \inf\{|x - y| : y \in E\}.$$

- (a) Show that D is a continuous function on \mathbb{R}^n (under the usual topology on \mathbb{R} and \mathbb{R}^n).
 (b) Show that $\{x \in \mathbb{R}^n : D(x) = |x|\}$ is closed in \mathbb{R}^n .

4. Let ω be a smooth 1-form on \mathbb{R}^2 that satisfies

$$\omega \wedge dx = -d(x^2) \wedge dy$$

and

$$\omega \wedge dy = dx \wedge d(y^2).$$

Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be the differentiable path that joins $(0, 0)$ to $(-1, 4)$ given by $\gamma(t) = (-\sin \frac{\pi}{2}t, 4t^3)$ for $t \in [0, 1]$. Compute $\int_{\gamma} \omega$.

5. a) Prove or provide a counterexample to the following statement: If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then there exists a real number L such that

$$\lim_{\epsilon \rightarrow 0} \int_{\epsilon \leq |x| \leq 1} \frac{f(x)}{x} dx = L$$

- b) What is your answer to part a) if there exist positive constants C and α such that for all $x \neq y \in \mathbb{R}$

$$|f(x) - f(y)| < C|x - y|^{\alpha} ?$$

Again, prove that the limit exists or give a counterexample.

6. Suppose that for each $j = 1, 2, \dots$, $g_j : [0, 1] \rightarrow \mathbb{R}$ is a continuous function such that $\int_0^1 |g_j(x)| dx \leq 1000$. Suppose $h : [0, 1] \rightarrow \mathbb{R}$ is a continuous function. Suppose that for each $n = 0, 1, 2, \dots$,

$$\lim_{j \rightarrow \infty} \int_0^1 x^n g_j(x) dx = \int_0^1 x^n h(x) dx .$$

Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Prove that

$$\lim_{j \rightarrow \infty} \int_0^1 f(x) g_j(x) dx = \int_0^1 f(x) h(x) dx .$$

7. Let $f = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $f_1(x_1, x_2) = x_1$ and for $x_2 \geq 0$

$$f_2(x_1, x_2) = \begin{cases} x_2 - x_1^2 & \text{if } x_2 \geq x_1^2 \\ \frac{x_2^2}{x_1^2} - x_2, & \text{if } x_1^2 \geq x_2 > 0 \\ 0 & \text{if } x_2 = 0 \end{cases}$$

If $x_2 < 0$, define f_2 by $f_2(x_1, x_2) = -f_2(x_1, -x_2)$. This function f is differentiable at $\vec{0}$, and you may use this fact without proving it, whenever needed, below.

- Show that f is differentiable (at all points in \mathbb{R}^2). Show that $f'(\vec{0}) = \text{identity}$.
- Prove that f is not one-to-one in any small neighborhood of the origin $\vec{0}$.
- State the inverse function theorem. In view of part b), the theorem does not apply to f near the origin $x = \vec{0}$. EXPLAIN. Explicitly what condition of the theorem is not met by the function f (at $\vec{0}$)?

8. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable.

- Assume that the Jacobian matrix $(\partial f_i / \partial x_j)$ has rank n everywhere. Prove that $f(\mathbb{R}^n)$ is open.
- Suppose that $f^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^n$ is compact. Prove that $f(\mathbb{R}^n)$ is closed.
- Assume that the Jacobian matrix $(\partial f_i / \partial x_j)$ has rank n everywhere, and that $f^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^n$ is compact. Prove that $f(\mathbb{R}^n) = \mathbb{R}^n$.

9. Define the function $g : [0, \infty) \rightarrow \mathbb{R}$ as follows:

$$g(x) = \int_0^x \frac{\sin^3 u}{u} du .$$

Note that this integral is well defined, since $|\sin u| \leq u$ for all $u > 0$. Prove that $\lim_{x \rightarrow \infty} g(x)$ exists in \mathbb{R} . (You don't have to find the limit.)

Name _____ ID number _____

Analysis Qualifying Exam, Spring 2002, Indiana University

Instructions. There are nine problems, each of equal value. Show your work, justifying all steps by direct calculation or by reference to an appropriate theorem. Good luck!

1. Let a_0, a_1, \dots, a_n be a set of real numbers satisfying

$$a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{n+1} = 0.$$

Prove that the polynomial $P_n(x) = a_0 + a_1x + \dots + a_nx^n$ has at least one root in $(0, 1)$.

2. Let $f_n : R \rightarrow R$ be differentiable, for all n , with derivative uniformly bounded (in absolute value) by 1. Further assume that $\lim_{n \rightarrow \infty} f_n(x) = g(x)$ exists for all $x \in R$. Prove that $g : R \rightarrow R$ is continuous.

3. Let $f : R^2 \rightarrow R$ have the property that for every $(x, y) \in R^2$, there exists *some* rectangular interval $[a, b] \times [c, d]$, $a < x < b$, $c < y < d$, on which f is Riemann integrable. Show that f is Riemann integrable on *any* rectangular interval $[e, f] \times [g, h]$.

4. Show that the sequence

$$1/2, (1/2)^{1/2}, ((1/2)^{1/2})^{1/2}, (((1/2)^{1/2})^{1/2})^{1/2}, \dots$$

converges to a limit L , and determine this limit.

5. Let $f, g : R^2 \rightarrow R$ be functions with continuous first derivative such that the map $F : (x, y) \rightarrow (f, g)$ has Jacobian determinant

$$\det \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

identically equal to one. Show that F is open, i.e., it takes open sets to open sets.

If also f is *linear*, i.e. f_x and f_y are constant, show that F is one-to-one.

6. Let $f : (0, 1] \rightarrow R$ have continuous first derivative, with $f(1) = 1$ and $|f'(x)| \leq x^{-1/2}$ if $|f(x)| \leq 3$. Prove that $\lim_{x \rightarrow 0^+} f(x)$ exists.

7. Letting $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in R^3 , evaluate the surface integral

$$F = - \int \int_S P(x, y, z) \nu \, dA,$$

where $\nu(x, y, z) = (x, y, z)$ denotes the outward normal to S , dA the standard surface element, and:

(a) $P(x, y, z) = P_0$, P_0 a constant.

(b) $P(x, y, z) = Gz$, G a constant.

Remark (not needed for solution): F corresponds to the total buoyant force exerted on the unit ball by an external, ideal fluid with pressure field P .

8. Compute the integral

$$\int_C y(z+1)dx + xzdy + xydz,$$

where $C : x = \cos \theta, y = \sin \theta, z = \sin^3 \theta + \cos^3 \theta, \quad 0 \leq \theta \leq 2\pi$.

9. Let X and Y be metric spaces and $f : X \rightarrow Y$. If $\lim_{p \rightarrow x} f(p)$ exists for all $x \in X$, show that $g(x) = \lim_{p \rightarrow x} f(p)$ is continuous on X .

1. In the classical *false position* method to find roots of $f(x) = 0$, one begins with two approximations x_0, x_1 and generates a sequence of (hopefully) better approximations via

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_0}{f(x_n) - f(x_0)} \quad \text{for } n = 1, 2, \dots$$

Consider the following sketch in which the function $f(x)$ is to be increasing and convex:

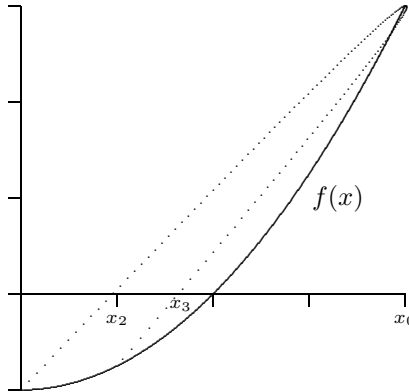


Fig. 1.2

The sequence $\{x_n\}$ is constructed as follows. We begin with the two approximations $(x_0, f(x_0))$ and $(x_1, f(x_1)) = (0, f(0))$. The chord is drawn between these two points; the point at which this chord crosses the x -axis is taken to be the next approximation x_2 . One then draws the chord between the two points $(x_0, f(x_0))$ and $(x_2, f(x_2))$. The next approximation x_3 is that point where this chord crosses the axis, as shown. For f strictly increasing and convex and for initial approximations $x_0 > 0, x_1 = 0$ with $f(x_0) > 0, f(x_1) < 0$, prove *rigorously* that this sequence must converge to the unique solution of $f(x) = 0$ over $[x_1, x_0]$.

2. (a) Show that it is possible to solve the equations

$$\begin{aligned} xu^2 + yzv + x^2z - 3 &= 0 \\ xyv^3 + 2zu - u^2v^2 - 2 &= 0 \end{aligned}$$

for (u, v) in terms of (x, y, z) in a neighborhood of $(1, 1, 1, 1, 1)$.

(b) Given that the inverse of the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

find $\frac{\partial u}{\partial x}$ at $(1, 1, 1)$.

3. Let X be a complete metric space and let Y be a subspace of X . Prove that Y is complete if and only if it is closed.

4. Suppose $f: K \rightarrow \mathbb{R}^1$ is a continuous function defined on a compact set K with the property that $f(x) > 0$ for all $x \in K$. Show that there exists a number $c > 0$ such that $f(x) \geq c$ for all $x \in K$.

5. Let $f(x)$ be a continuous function on $[0, 1]$ which satisfies

$$\int_0^1 x^n f(x) dx = 0 \quad \text{for all } n = 0, 1, \dots$$

Prove that $f(x) = 0$ for all $x \in [0, 1]$.

6. Show that the Riemann integral $\int_0^\infty \frac{\sin x}{x} dx$ exists.

7. Let

$$G(x, y) = \begin{cases} x(1-y) & \text{if } 0 \leq x \leq y \leq 1 \\ y(1-x) & \text{if } 0 \leq y \leq x \leq 1 \end{cases}$$

Let $\{f_n(x)\}$ be a uniformly bounded sequence of continuous functions on $[0, 1]$ and consider the sequence

$$u_n(x) = \int_0^1 G(x, y) f_n(y) dy.$$

Show that the sequence $\{u_n(x)\}$ contains a uniformly convergent subsequence on $[0, 1]$.

8. Let f be a real-valued function defined on an open set $U \subset \mathbb{R}^2$ whose partial derivatives exist everywhere on U and are bounded. Show that f is continuous on U .

9. For $x \in \mathbb{R}^3$ consider spherical coordinates $x = r\omega$ where $|\omega| = 1$ and $|x| = r$. Let ω_k be the k 'th component of ω for any $k = 1, 2, 3$. Use the divergence theorem to evaluate the surface integral

$$\int_{|\omega|=1} \omega_k dS.$$

10. Let $\{f_k\}$ be a sequence of continuous functions defined on $[a, b]$. Show that if $\{f_k\}$ converges uniformly on (a, b) , then it also converges uniformly on $[a, b]$.

11. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a continuous mapping. Show that $f(S)$ is bounded in \mathbb{R}^k if S is a bounded set in \mathbb{R}^n .

1. In the classical *false position* method to find roots of $f(x) = 0$, one begins with two approximations x_0, x_1 and generates a sequence of (hopefully) better approximations via

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Consider the following sketch in which the function $f(x)$ is to be increasing and convex:

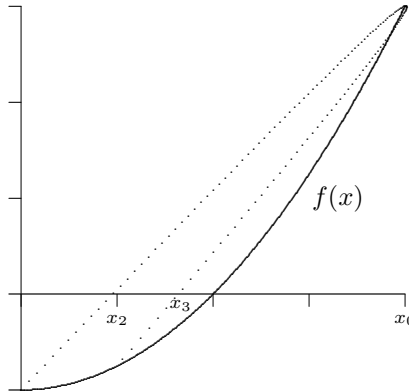


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for (u, v) in terms of (x, y, z) in a neighborhood of $(1, 1, 1, 1, 1)$.

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find $\frac{\partial u}{\partial x}$ at $(1, 1, 1)$.

3. Let X be a complete metric space and let Y be a subspace of X . Prove that Y is complete if and only if it is closed.

4. Suppose $f: K \rightarrow \mathbb{R}^1$ is a continuous function defined on a compact set K with the property that $f(x) > 0$ for all $x \in K$. Show that there exists a number $c > 0$ such that $f(x) \geq c$ for all $x \in K$.

5. Let $f(x)$ be a continuous function on $[0, 1]$ which satisfies

$$\int_0^1 x^n f(x) dx = 0 \quad \text{for all } n = 0, 1, \dots$$

Prove that $f(x) = 0$ for all $x \in [0, 1]$.

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Tier 1 Analysis Exam
January 2003

1. Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$. Which of the following statements is equivalent to the continuity of f at 0? (Provide justification for each of your answers.)
 - a) For every $\varepsilon \geq 0$ there exists $\delta > 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| \leq \varepsilon$.
 - b) For every $\varepsilon > 0$ there exists $\delta \geq 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| \leq \varepsilon$.
 - c) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x| \leq \delta$ implies $|f(x) - f(0)| \leq \varepsilon$.
2. Consider a uniformly continuous real-valued function f defined on the interval $[0, 1)$. Show that $\lim_{t \rightarrow 1^-} f(t)$ exists. Is a similar statement true if $[0, 1)$ is replaced by $[0, \infty)$?
3. Let f be a real-valued continuous function on $[0, 1]$ such that $f(0) = f(1)$. Show that there exists $x \in [0, 1/2]$ such that $f(x) = f(x + 1/2)$.
4. If f is differentiable on $[0, 1]$ with continuous derivative f' , show that

$$\int_0^1 |f(x)| dx \leq \max \left\{ \left| \int_0^1 f(x) dx \right|, \int_0^1 |f'(x)| dx \right\}$$

5. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous and with compact support, i.e. there exists $R > 0$ such that $f(x, y) = 0$ if $x^2 + y^2 \geq R^2$.
 - a) Show that the integral

$$g(u, v) = \iint_{\mathbb{R}^2} \frac{f(x, y)}{\sqrt{(x-u)^2 + (y-v)^2}} dx dy$$

converges for all $(u, v) \in \mathbb{R}^2$, and show that $g(u, v)$ is continuous in (u, v) .

- b) Show that, if in addition f has continuous first order partial derivatives, then so does g and

$$\frac{\partial g}{\partial u}(u, v) = \iint_{\mathbb{R}^2} \frac{\frac{\partial f}{\partial x}(x, y)}{\sqrt{(x-u)^2 + (y-v)^2}} dx dy .$$

6. Show that for any two functions f, g which have continuous second order partial derivatives, defined in a neighborhood of the sphere $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in \mathbb{R}^3 , one has

$$\int_S (\nabla f \times \nabla g) \cdot d\mathbf{S} = 0$$

where $\nabla f, \nabla g$ are the gradient of f, g respectively.

7. Show that if $\{x_n\}$ is a bounded sequence of real numbers such that $2x_n \leq x_{n+1} + x_{n-1}$ for all n , then $\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = 0$.

8. For a non-empty set X , let \mathbb{R}^X be the set of all maps from X to \mathbb{R} . For $f, g \in \mathbb{R}^X$, define

$$d(f, g) = \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$

a) Show that (\mathbb{R}^X, d) is a metric space.

b) Show that $f_n \rightarrow f$ in (\mathbb{R}^X, d) if and only if f_n converges uniformly to f .

9. Show that if $f : [0, 1] \rightarrow \mathbb{R}$ is continuous, and $\int_0^1 f(x)x^{2n}dx = 0$, $n = 0, 1, 2, \dots$ then $f(x) = 0$ for all $x \in [0, 1]$.

10. a) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function. Show that for any $x, y \in \mathbb{R}^n$, there exists $z \in \mathbb{R}^n$ such that

$$f(x) - f(y) = Df(z) \cdot (x - y)$$

where $Df(z)$ denotes the derivative matrix of f (in this case it is the same as the gradient of f) at z , and “ \cdot ” denotes the usual dot product in \mathbb{R}^n .

b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a differentiable map. Show that if f has the property that $\|Df(z) - I\| < \frac{1}{2n}$ for all $z \in \mathbb{R}^n$, where I is the $n \times n$ identity matrix, then f is a diffeomorphism, i.e. f is one-to-one, onto and f^{-1} is also differentiable. (For a matrix $A = (a_{ij})$, $\|A\| = (\sum_{i,j} a_{ij}^2)^{1/2}$.)

Tier I exam in analysis - August 2003

Answer all the problems. Justify your answers.

1. Let $f(x)$ be a function that is continuous in $[-1, 1]$, differentiable in $(-1, 1)$, and satisfies $f(-1) = -\pi/2$, $f(1) = \pi/2$, $f'(x) \geq \frac{1}{\sqrt{1-x^2}}$ in $(-1, 1)$. Prove that $f(x) = \arcsin(x)$ in $[-1, 1]$.

2. Determine the values of $x \in \mathbf{R}$ such that the series

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n+x^2)}}$$

converges.

3. Recall that a square matrix M is called orthogonal if its rows form an orthonormal set. The set of all orthogonal matrices will be denoted by O . (Note that an orthogonal matrix necessarily satisfies the condition $MM^t = I$ where M^t denotes the transpose of M .)

$$\text{Let } M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a given element of O where a, b, c and d are real numbers.

- (i). Prove that, except for 4 special matrices M_0 , there always exists a number $\delta > 0$ and three functions f, g, h , continuously differentiable for $x \in (a - \delta, a + \delta)$, such that

$$\begin{pmatrix} x & f(x) \\ g(x) & h(x) \end{pmatrix} \in O,$$

for all $x \in (a - \delta, a + \delta)$ with $f(a) = b$, $g(a) = c$ and $h(a) = d$.

- (ii). What are the four exceptional matrices of part (i)?

4. A vector field $\vec{F} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is said to be conservative in an open set D if the line integral $\int_C \vec{F} \cdot ds = 0$ for every closed curve $C \subset D$. Find all numbers a and b such that the vector field

$$\vec{F}(x, y) = \left(\frac{x + ay}{x^2 + y^2}, \frac{bx + y}{x^2 + y^2} \right)$$

is conservative in

$$D = \{(x, y) : \frac{1}{9} < x^2 + y^2 < \frac{1}{4}\}.$$

5. Consider the triangle with vertices $(3, 0)$, $(5, 0)$ and $(5, 1)$ in the (x, y) -plane. Revolve it around the y -axis in (x, y, z) -space \mathbf{R}^3 to sweep out a "triangular torus" T , evaluate the surface integral

$$\int_T \vec{v} \cdot \vec{n} \, dS.$$

Here $\vec{v} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is the vector field $\vec{v}(x, y, z) = (-y, x, z)$, \vec{n} is the outward unit normal field on T , and dS is the usual surface-area element on T .

6. Suppose $f: \mathbf{R}^2 \rightarrow \mathbf{R}$ is a C^∞ function that has a critical point at $(0, 0)$. Suppose that $f(0, 0) = 0$ and that all the first and second order partial derivatives of f vanish at $(0, 0)$. Also, assume that not all third order partial derivatives vanish at $(0, 0)$. Show that f can have neither a local max nor a local min at the critical point $(0, 0)$.
7. Recall that a function $g: [a, b] \rightarrow \mathbf{R}$ is said to be Lipschitz if there is a constant K such that $|g(x) - g(y)| \leq K |x - y|$ for all $x, y \in [a, b]$.

Assume that f is a bounded Riemann integrable function on $[a, b]$. Prove that for each $\varepsilon > 0$, there exists a Lipschitz function g such that

$$\int_a^b |f(x) - g(x)| dx < \varepsilon.$$

8. (a) If $B \subset \mathbf{R}^n$ is a bounded set and $f: B \rightarrow \mathbf{R}$ is uniformly continuous, show that $f(B)$ is bounded.
- (b) Give an example to show that the conclusion of part (a) is not necessarily true if f is merely continuous on B .

TIER 1 Analysis Exam January 2004

Instruction: Solve as many of these problems as you can. Be sure to justify all your answers.

1. Let $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be strictly increasing, integer valued, sequences. Show that if for each integer $n \geq 1$,

$$p_n \cdot q_{n-1} - p_{n-1} \cdot q_n = 1,$$

then the sequence of quotients p_n/q_n converges.

2. Consider the following system of equations

$$x \cdot e^y = u,$$

$$y \cdot e^x = v.$$

- (a) Show that there exists an $\epsilon > 0$ such that given any u and v with $|u| < \epsilon$ and $|v| < \epsilon$, the above system has a unique solution $(x, y) \in \mathbb{R}^2$.
 (b) Exhibit a pair $(u, v) \in \mathbb{R}^2$ such that there exist two distinct solutions to this system. Justify your answer.
3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $f'(a) < f'(b)$ for some $a < b$. Prove that for any $z \in (f'(a), f'(b))$, there is a $c \in (a, b)$ such that $f'(c) = z$. Note: The derivative function f' may not be continuous.

4. Let $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be continuously differentiable. Let $Df(x)$ denote the differential (or derivative) of f at the point $x \in \mathbb{R}^4$. Prove or provide a counter-example: The set of points x where $Df(x)$ has a null space of dimension 2 or greater is closed in \mathbb{R}^4 .

5. Let $C([0, 1])$ denote the collection of continuous real valued functions on $[0, 1]$. Define $\Phi : C([0, 1]) \rightarrow C([0, 1])$ by

$$[\Phi(f)](t) = 1 + \int_0^t s^2 e^{-f(s)} ds \quad t \in [0, 1]$$

for $f \in C([0, 1])$. Define $f_0 \in C([0, 1])$ by $f_0 \equiv 1$ (i.e. the function of constant value 1). Let $f_n = \Phi(f_{n-1})$ for $n = 1, 2, \dots$.

- (a) Prove that $1 \leq f_n(t) \leq 1 + 1/3$ for all $t \in [0, 1]$ and $n = 1, 2, \dots$
 (b) Prove that

$$|f_{n+1}(x) - f_n(x)| < \frac{1}{3} \sup_{t \in [0, 1]} |f_n(t) - f_{n-1}(t)|$$

for all $x \in [0, 1]$ and for $n = 1, 2, \dots$. Hint: Show that $|e^{-(x+\delta)} - e^{-x}| < \delta$ for $x > 0$ and $\delta 0$.

- (c) Show that the sequence of functions $\{f_n\}$ converges uniformly to some function $f \in C([0, 1])$. Be sure to indicate any theorems that you use.

6. Let I be a closed interval in \mathbb{R} , and let f be a differentiable real valued function on I , with $f(I) \subset I$. Suppose $|f'(t)| < 3/4$ for all $t \in I$. Let x_0 be any point in I and define a sequence x_n by $x_{n+1} = f(x_n)$ for every $n > 0$. Show that there exists $x \in I$ with $f(x) = x$ and $\lim x_n = x$.

7. Let

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y} & \text{if } x^2 + y \neq 0, \\ 0 & \text{if } x^2 + y = 0. \end{cases}$$

- (a) Show that f has a directional derivative (in every direction) at $(0, 0)$, and show that f is not continuous at $(0, 0)$.
- (b) Prove or provide a counterexample: If $P_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $P_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are any two functions such that $P_1(0, 0) = (0, 0) = P_2(0, 0)$, and such that $f \circ P_i$ is differentiable at $(0, 0)$, with nonvanishing derivative at $(0, 0)$ for $i = 1, 2$, then $f \circ (P_1 + P_2)$ is differentiable at $(0, 0)$.
8. Let $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ be the unit ball. Let $v = (v_1, v_2, v_3)$ be a smooth vector field on B , which vanishes on the boundary ∂B of B and satisfies

$$\operatorname{div} v(x, y, z) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0, \quad \forall (x, y, z) \in B.$$

Prove that

$$\int_B x^n v_1(x, y, z) dx dy dz = 0, \quad \forall n = 0, 1, 2, \dots$$

9. Suppose that $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function on $[0, 1]$ with

$$\int_0^1 f(x) dx = \int_0^1 f(x)(x^n + x^{n+2}) dx$$

for all $n = 0, 1, 2, \dots$. Show that $f \equiv 0$.

10. Suppose that $f : [0, 1] \rightarrow \mathbb{R}$ has a continuous second derivative, $f(0) = f(1) = 0$, and $f(x) > 0$ for all $x \in (0, 1)$. Prove that

$$\int_0^1 \left| \frac{f''(x)}{f(x)} \right| dx > 4.$$

Tier I Analysis Exam-August 2004

1. (A) Suppose A and B are nonempty, disjoint subsets of \mathbb{R}^n such that A is compact and B is closed. Prove that there exists a pair of points $a \in A$ and $b \in B$ such that

$$\forall x \in A, \forall y \in B, \quad \|x - y\| \geq \|a - b\|.$$

Prove this fact from basic principles and results; do not simply cite a similar or more general theorem. Here and in what follows, $\|\cdot\|$ denotes the usual Euclidean norm: for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\|x\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$.

(B) Suppose that in problem (A) above, the assumption that the set A is compact is replaced by the assumption that A is closed. Does the result still hold? Justify your answer with a proof or counterexample.

2. (A) Prove the following classic result of Cauchy: Suppose $r(1), r(2), r(3), \dots$ is a monotonically decreasing sequence of positive numbers. Then $\sum_{k=1}^{\infty} r(k) < \infty$ if and only if $\sum_{n=1}^{\infty} 2^n r(2^n) < \infty$.

(B) Use the result in part (A) to prove the following theorem: Suppose a_1, a_2, a_3, \dots is a monotonically decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n = \infty$. For each $n \geq 1$, define the positive number $c_n = \min\{a_n, 1/n\}$. Then $\sum_{n=1}^{\infty} c_n = \infty$.

3. Suppose $g : [0, \infty) \rightarrow [0, 1]$ is a continuous, monotonically increasing function such that $g(0) = 0$ and $\lim_{x \rightarrow \infty} g(x) = 1$.

Suppose that for each $n = 1, 2, 3, \dots$, $f_n : [0, \infty) \rightarrow [0, 1]$ is a monotonically increasing (but not necessarily continuous) function. Suppose that for all $x \in [0, \infty)$, $\lim_{n \rightarrow \infty} f_n(x) = g(x)$. Prove that $f_n \rightarrow g$ uniformly on $[0, \infty)$ as $n \rightarrow \infty$.

4. Let $x \in \mathbb{R}^3$ and let $f(x) \in C^1(\mathbb{R}^3)$. Further let $n = x/\|x\|$ for $x \neq 0$. Show that the surface integral

$$I \equiv \int_{\|x\|=1} f(x) dS_x$$

can be expressed in the form of a volume integral

$$I = \int_{\|x\|<1} \left(\frac{2}{\|x\|} f(x) + n \cdot \nabla f(x) \right) dx.$$

Hint: Write the integrand in I as $n \cdot (nf)$.

5. Let $x_0 \in \mathbb{R}$ and consider the sequence defined by

$$x_{n+1} = \cos(x_n) \quad (n = 0, 1, \dots)$$

Prove that $\{x_n\}$ converges for arbitrary x_0 .

6. Let $\alpha > 0$ and consider the integral

$$J_\alpha = \int_0^\infty \frac{e^{-x}}{1 + \alpha x} dx.$$

Show that there is a constant c such that

$$\alpha^{1/2} J_\alpha \leq c.$$

7. Consider the infinite series

$$\sum_{n=1}^{\infty} X_n(x) T_n(t)$$

where (x, t) varies over a rectangle $\Omega = [a, b] \times [0, \tau]$ in \mathbb{R}^2 . Assume that

- (i) The series $\sum_{n=1}^{\infty} X_n(x)$ converges uniformly with respect to $x \in [a, b]$;
- (ii) There exists a positive constant c such that $|T_n(t)| \leq c$ for every positive integer n and every $t \in [0, \tau]$;
- (iii) For every t such that $t \in [0, \tau]$, $T_1(t) \leq T_2(t) \leq T_3(t) \leq \dots$

Prove that $\sum_{n=1}^{\infty} X_n(x) T_n(t)$ converges uniformly with respect to both variables together on Ω .

Hint: Let $S_N = \sum_{n=1}^N X_n(x) T_n(t)$, $s_N = \sum_{n=1}^N X_n(x)$. For $m > n$ find an expression for $S_m - S_n$ involving $(s_k - s_n)$ for an appropriate range of values of k .

8. Let $v(x) \in C^\infty(\mathbb{R})$ and assume that for each γ in a neighborhood of the origin there exists a function $u(x, v, \gamma)$ which is C^∞ in x such that

$$\gamma \frac{\partial}{\partial x}(u + v) = \sin(u - v).$$

Assuming that

$$u = u_0 + \gamma u_1 + \gamma^2 u_2 + \gamma^3 u_3 + \dots$$

where $u_0(0) = v(0)$ and for all n the u_n 's are functions of v but are independent of γ , find u_0 , u_1 , u_2 and u_3 .

9. All partial derivatives $\partial^{m+n} f / \partial x^m \partial y^n$ of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ exist everywhere. Does it imply that f is continuous? Prove or give a counterexample.

10. Decide whether the two equations

$$\sin(x + z) + \ln(yz^2) = 0, \quad e^{x+z} + yz = 0,$$

implicitly define (x, y) near $(1, 1)$ as a function of z near -1 .

Tier I exam in analysis - January 2005

Solve all problems. Justify your answers in detail. The exam's duration is 3 hours

1. Define

$$S = \{(x, y, z) \in \mathbb{R}^3, \quad x^2 + 2y^2 + 3z^2 = 1\}, \quad f(x, y, z) = x + y + z.$$

- a. Prove that S is a compact set.
b. Find the maximum and minimum of f on S .

2. Let $g : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be a continuous function, and define functions $f_n : [0, 1] \rightarrow \mathbb{R}$ by

$$f_n(x) = \int_0^1 g(x, y)y^n dy \quad x \in [0, 1], n = 1, 2, \dots$$

Show that the sequence $(f_n)_{n=1}^\infty$ has a subsequence which converges uniformly on $[0, 1]$.

3. Consider the subset $H = \{(a, b, c, d, e)\}$ of \mathbb{R}^5 such that the polynomial

$$ax^4 + bx^3 + cx^2 + dx + e$$

has at least one real root.

- a. Prove that $(1, 2, -4, 3, -2)$ is an interior point of H
b. Find a point in H that is *not* an interior point. Justify your claim.
4. Consider a twice differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$, a number $a \in \mathbb{R}$, and $h > 0$. Show that there exists a point $c \in \mathbb{R}$ such that

$$f(a) - 2f(a + h) + f(a + 2h) = h^2 f''(c).$$

5. Prove or give a counterexample: If $f(x)$ is differentiable for every $x \in \mathbb{R}$, and if $f'(0) = 1$, then there exists $\delta > 0$ such that $f(x)$ is increasing on $(-\delta, \delta)$.

6. Let $f(x)$ be a bounded function on $(0, 2)$. Suppose that for every $x, y \in (0, 2)$, $x \neq y$, there exists $z \in (0, 2)$ such that

$$f(x) - f(y) = f(z)(x - y).$$

- a. Show that f need not be a differentiable function.
b. Suppose that such a z can always be found between x and y . Show that f is twice differentiable.
7. Consider the torus

$$T = \{x = (a + r \sin u) \cos v, \quad y = (a + r \sin u) \sin v, \quad z = r \cos u,$$

$$0 \leq r \leq b, \quad 0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2\pi\},$$

where $a > b$. Find the volume and surface area of T .

8. Let Ω be a bounded subset of R^n , and $f : \Omega \rightarrow R^n$ a uniformly continuous function. Show that f must be bounded.

Outline of Solutions:

- a. It suffices to show that S is closed and bounded. Closeness follows since $S = \{h^{-1}(1)\}$, for a continuous function h . Boundedness follows since clearly S is contained in the cube $[-1, 1]^3$.
b. Both maximum and minimum are obtained at internal points on S , and can therefore be found by the Lagrange method. The Lagrange equations imply at once that $\lambda \neq 0$, and $\frac{1}{2\lambda} = x = 2y = 3z$. Solving from S we find that the maximal value is $\sqrt{11/6}$, and the minimal value is its negative.

- $f_n(0) = 0$, and the functions f_n are equicontinuous because

$$|f_n(x) - f_n(x')| \leq \sup_y |g(x, y) - g(x', y)|,$$

and this quantity tends to zero as $|x - x'| \rightarrow 0$ by the continuity of g . This Arzela-Ascoli applies.

- Write the polynomial $x^4 + 2x^3 - 4x^2 + 3x - 2$. Obviously $x = 1$ is a root, so the triplet is indeed in H .

Define the function $F(a, b, c, d, e, f, x) = ax^4 + bx^3 + cx^2 + ed + f$. Clearly $F(1, 2, -4, 3, -2, 1) = 0$, while $F_x = 5 \neq 0$ at that point. Therefore there exists an open neighborhood U of $(1, 2, -4, 3, -2)$ and a C^1 function g such that for all points (a, b, c, d, e) in U we have $F(a, b, c, d, e, g(a, b, c, d, e)) = 0$.

Clearly $(0, 0, 1, 0, 0)$ is in H . But the the points $(0, 0, 1, 0, \mu^2)$ are not in the set for $\mu \neq 0$ (Since $x^2 + \mu^2$ has no real root).

- Apply the mean-value theorem to the function $F(x) = f(x + h) - f(x)$ to get

$$f(a) - 2f(a + h) + f(a + 2h) = F(a + h) - F(a) = hF'(d) = h(f'(d + h) - f'(d))$$

for some d , then apply MVT again to the right-hand side.

- Counterexample - $f(x) = x + 2x^2 \sin(1/x)$.

- a. Let $f = x$ for $0 \leq x \leq 1$, and $f = 1$ for $1 \leq x \leq 2$.

Since f is bounded, $\lim_{y \rightarrow x} f(y) = f(x)$. Furthermore, $\lim_{x \rightarrow y} \frac{f(y) - f(x)}{x - y} = f'(y)$. Therefore f is differentiable. Also, the last identity implies $f' = f$, thus $f(x) = ce^x$.

- The Jacobian is given by $J = r(a + \sin u)$, and hence $V = 2\pi^2 ab^2$. Observing that the boundary is given by $r = b$, a simple computation gives $\|N\| = \|T_u \times T_v\| = b(a + b \sin u)$. Therefore $S = 4\pi^2 ab$. Of course, it is also possible to solve with the slice method.

- Choose $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Assume that f is not bounded, and choose $x_k \in \Omega$ such that $|f(x_{k+1})| > |f(x_k)| + 1$ for all k . Observe that $|f(x_j) - f(x_k)| > 1$ whenever $j \neq k$. However, by Bolzano-Weierstrass, we must have $|x_j - x_k| < \delta$ for some $j \neq k$, which gives a contradiction.

Tier I Analysis Exam
August, 2005

Justify your answers. All problems carry equal weight.

1. Let $p > 0$. Evaluate

$$\lim_{n \rightarrow \infty} \frac{1^p + 3^p + \dots + (2n-1)^p}{n^{p+1}}$$

2. For $x > 0$, define

$$\phi(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \end{cases}$$

where p, q have no common factor and $q \geq 1$.

- (a) Where is ϕ continuous?
(b) Where is ϕ differentiable?

3. Let a, b be real with $|b| > \max\{1, |a|\}$. For $x \in \mathbf{R}$, define

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(a^n x)}{b^n}$$

- (a) Show that f is uniformly continuous on all of \mathbf{R} .
(b) Let

$$\gamma = \{(x, y) : y = f(x), 0 \leq x \leq 1\}$$

be the graph of f over the unit interval. Show that γ has finite length.

4. Let M_2 denote the set of 2-by-2 matrices with real entries, and for $A \in M_2$, define $S(A) = A^2$. Does the mapping $S : M_2 \rightarrow M_2$ have a local inverse near the identity matrix?
5. Fix $a > 0$. Let x_1, \dots, x_n be non-negative numbers with

$$\sum_{i=1}^n x_i = na.$$

Show that

$$\sum_{i < j} x_i x_j \leq \frac{1}{2} n(n-1) a^2.$$

Please turn over.

6. Let p be real. Suppose $f : \mathbf{R}^n - \{0\} \rightarrow \mathbf{R}$ is continuously differentiable, and satisfies

$$f(\lambda x) = \lambda^p f(x), \text{ for all } x \neq 0 \text{ and for all } \lambda > 0.$$

Let $\nabla f(x)$ denote the gradient of f at x and \cdot the Euclidean inner product. Prove that

$$x \cdot \nabla f(x) = p f(x), \quad x \neq 0.$$

7. A family \mathcal{F} of continuous real-valued functions of a real variable is called *equicontinuous at x* if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $f \in \mathcal{F}$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

\mathcal{F} is called *equicontinuous on the set E* if it is equicontinuous at each point x of E . (Note: the constant δ may depend on both ϵ and x .)

Now suppose \mathcal{F} is a family of continuous real-valued functions defined on an open interval $I \subseteq \mathbf{R}$, and let $x_0 \in I$.

- (a) Suppose \mathcal{F} is equicontinuous at every point of $I \setminus \{x_0\}$. Must \mathcal{F} also be equicontinuous at x_0 ?
- (b) Suppose \mathcal{F} is equicontinuous at x_0 . Must \mathcal{F} also be equicontinuous at every point in some neighborhood J of x_0 ?
8. Let U be an open subset of \mathbf{R}^n and $f : U \rightarrow \mathbf{R}^n$ be differentiable. Suppose there exists $C > 0$ such that

$$|f(x) - f(y)| \geq C|x - y|$$

for all $x, y \in U$. Let $df(x)$ denote the Jacobian derivative of f at x (that is, the linear mapping given by the n by n matrix of partial derivatives). Show that $\det df(x) \neq 0$ for all $x \in U$.

9. Let $m > 0$ be a real number, let $r = (x^2 + y^2 + z^2)^{1/2}$, and consider the vector field on \mathbf{R}^3 given by $\vec{F} = r^m \cdot (x, y, z) = (x^2 + y^2 + z^2)^{m/2}(x, y, z)$.

- (a) Compute the divergence $\text{div}(\vec{F})$.
- (b) Using part (a) and the Divergence Theorem, calculate

$$\iiint_{B^3} r^m dV$$

where $B^3 = \{(x, y, z) : r \leq 1\}$ is the closed unit ball centered at the origin and $dV = dx dy dz$ is the Euclidean volume.

End of exam.

Analysis Tier I Exam January 2006

All questions are worth 10 points. In question 7, each part is worth 5 points.

1. Show that the function given by

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable for all $x \in \mathbb{R}$, but not *continuously* differentiable at $x = 0$.

2. Let $\{f_n\}$ be the sequence of functions given by

$$f_n(x) = nxe^{-nx}.$$

Prove that $\{f_n\}$ converges to 0 pointwise but not uniformly on the interval $[0, 1]$ as $n \rightarrow \infty$.

3. Let n be a positive integer and define f on $[0, \infty)$ by $f(x) = \sqrt[n]{x}$. Give a direct ϵ, δ proof that f is continuous on $[0, \infty)$.
4. Associating any 2×2 real matrix (a_{ij}) with a point $(a_{11}, a_{12}, a_{21}, a_{22}) \in \mathbb{R}^4$, prove that the set of all invertible, real matrices is not a connected set in \mathbb{R}^4 .
5. Define a sequence $\{r_n\}$ by $r_0 = 1$, and $r_{n+1} = (2/3)r_n + 1$ for $n \geq 0$. Let the sequence $\{c_n\}$ be defined by $c_0 = 1/4$, and

$$c_{n+1} = \frac{r_{n+1}\sqrt{c_n}}{3}$$

for $n \geq 0$. Prove that

$$\lim_{n \rightarrow \infty} c_n \text{ exists}$$

and determine what the limit is.

Hint: First argue that $\{r_n\}$ converges.

6. Do there exist continuous functions $f(x, y)$ and $g(x, y)$ in a neighborhood of $(0, 1)$ such that $f(0, 1) = 1$ and $g(0, 1) = -1$ and such that

$$\begin{aligned} [f(x, y)]^3 + xg(x, y) - y &= 0, \\ [g(x, y)]^3 + yf(x, y) - x &= 0? \end{aligned}$$

Justify your answer.

7. Let $\epsilon > 0$ and a positive integer n be given. Let $F \subset \mathbb{Z} \times \mathbb{Z}$ be defined by $F = \{(i, j) : 1 \leq i < j \leq n\}$ and let E be any subset of F . Then define a real-valued function $G_{E, \epsilon}$ on \mathbb{R}^n by

$$G_{E, \epsilon}(x_1, \dots, x_n) = (n + \epsilon) \sum_{j=1}^n \sin^2 x_j - \sum_{(i, j) \in E} (x_i - x_j)^2.$$

- a. Take $n = 3$ and $E = F$. Show that $G_{F, \epsilon}$ has a local minimum at the origin.
- b. For arbitrary positive integer n and E any subset of F , show that $G_{E, \epsilon}$ has a local minimum at the origin.
8. Let $D \subset \mathbb{R}^2$ be an arbitrary bounded open set with C^1 boundary whose perimeter P is finite. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a given C^1 function satisfying the condition

$$|f(x, y)| \leq 1 \quad \text{for all } (x, y) \in D.$$

Establish the inequality

$$\left| \iint_D \frac{\partial f}{\partial y}(x, y) dx dy \right| \leq P.$$

9. Suppose $K > 0$, and $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a differentiable mapping with $|\partial F_i / \partial x_j| < K$ at every point, for every $1 \leq i, j \leq 2$. Show that there exists $C > 0$ such that F satisfies the Lipschitz condition

$$\|F(p) - F(q)\| \leq C \|p - q\| \quad \text{for all } p, q \in \mathbb{R}^2.$$

Here $\|p - q\|$ denotes the usual Euclidean distance between p and q in \mathbb{R}^2 .

10. A family \mathcal{F} of functions is said to be *uniformly equicontinuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $g \in \mathcal{F}$,

$$|x_1 - x_2| < \delta \Rightarrow |g(x_1) - g(x_2)| < \epsilon.$$

Note: δ does not depend on g or x_1 or x_2 . Now suppose that $f : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ is a bounded continuous function. For each $y \in [0, 1]$, define $g_y : \mathbb{R} \rightarrow \mathbb{R}$ by $g_y(x) = f(x, y)$. Suppose that for each y we know that

$$\lim_{x \rightarrow \infty} g_y(x) = 0 = \lim_{x \rightarrow -\infty} g_y(x).$$

Must any such family $\mathcal{F} := \{g_y : 0 \leq y \leq 1\}$ be uniformly equicontinuous? If so, prove it. If not, provide a counter-example.

Tier 1 Analysis Exam

August 2006

- Let $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function.
 - Prove that if f is uniformly continuous, and if $\{p_k\}$ is a Cauchy sequence in A , then $\{f(p_k)\}$ is a Cauchy sequence in \mathbb{R}^m .
 - Give an example of continuous f and a Cauchy sequence $\{p_k\}$ in some A (you may take $n = m = 1$) for which $\{f(p_k)\}$ is not a Cauchy sequence.
- Let $f : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Assume f is differentiable everywhere in (a, b) , except possibly at a point c . Show that, if $\lim_{x \rightarrow c} f'(x)$ exists and is equal to L , then f is differentiable at c and $f'(c) = L$.
- Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 function with $g(\frac{1}{3}, \frac{2}{3}) = 3$, $\frac{\partial g}{\partial r}(\frac{1}{3}, \frac{2}{3}) = -1$, and $\frac{\partial g}{\partial s}(\frac{1}{3}, \frac{2}{3}) = -4$, where (r, s) are the coordinates for the \mathbb{R}^2 . Define $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$f(x, y, z) = g\left(\frac{x}{z}, \frac{y}{z}\right),$$

for $z \neq 0$. Show that the level surface $f^{-1}(3)$ has a tangent plane at the point $(1, 2, 3)$ and find a linear equation for it.

- For which positive integers k does the series

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/k)}{n}$$

converge? Justify your answer with a proof.

- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Let $x \in \mathbb{R}$ and define the sequence $\{x_n\}$ inductively by setting $x_0 = x$ and $x_{n+1} = f(x_n)$. Suppose that $\{x_n\}$ is bounded. Prove that there exists $y \in \mathbb{R}$ such that $f(y) = y$.
- Decide whether or not the function $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{x}{q}, & \text{if } x \notin \mathbb{Q} \text{ and } y = \frac{p}{q} \in \mathbb{Q} \\ \frac{y}{q}, & \text{if } y \notin \mathbb{Q} \text{ and } x = \frac{p}{q} \in \mathbb{Q} \\ 0, & \text{if } (x, y) \in \mathbb{Q} \times \mathbb{Q} \text{ or } (x, y) \in \mathbb{R} \setminus \mathbb{Q} \times \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is Riemann integrable on $[0, 1] \times [0, 1]$. Prove your decision from the definition without invoking any theorems about integrable functions. (Here all fractions $\frac{p}{q}$ are assumed to be reduced.)

7. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 function (i.e. f has continuous second order partial derivatives). Suppose $p_0 \in \mathbb{R}^n$ is a critical point of f . If

$$\det \left[\frac{\partial^2 f}{\partial x_i \partial x_j} (p_0) \right] \neq 0,$$

show that p_0 is isolated, i.e. there is a neighborhood of p_0 in which p_0 is the only critical point of f .

8. Prove that $\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ converges pointwise but not uniformly on \mathbb{R} . Let $f(x) = \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$.

Is it true that the Riemann integral $\int_0^1 f(x) dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x}{1+n^2x^2} dx$? Justify.

9. Let $\{a_n\}$ be a sequence. Show that

$$\limsup_{n \rightarrow \infty} \frac{a_n}{n} \leq \limsup_{n \rightarrow \infty} (a_n - a_{n-1}).$$

10. Let $Q \subset \mathbb{R}^3$ be any solid rectangular box with one vertex at the origin. Show that

$$\int_{\partial Q} \frac{\mathbf{x} \cdot \hat{\mathbf{n}}}{\|\mathbf{x}\|^3} dS = \frac{\pi}{2}.$$

Here $\hat{\mathbf{n}}$ is the unit outer normal on ∂Q and dS is the area element. (You should notice that this integral is not an improper integral.)

Tier I Analysis Exam January 2007

Notations: \mathbb{R}^n denotes the n -dimensional Euclidean space with the standard scalar (inner) product $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$ and the Euclidean norm $|x| = \sqrt{\langle x, x \rangle}$.

1. Use ε - δ notation to state the condition that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is not continuous on \mathbb{R} .

2. For $A > 0$ consider the sequence $x_{n+1} = \frac{1}{2} \left(x_n + \frac{A}{x_n} \right)$ ($n = 1, 2, \dots$) with $x_1 > 0$. Show that $\{x_n\}$ converges and find its limit.

3. Let $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and $X = \{f \in C(\Omega) : |f(x) - f(y)| \leq |x - y| \quad \forall x, y \in \Omega, |f(0)| < 1\}$ where $C(\Omega)$ denotes the space of continuous functions f from Ω to \mathbb{R} (both with standard, Euclidean topologies), with sup-norm $\|f\| := \sup_{X \in \Omega} |f(x)|$. Show that X is a sequentially compact subset of $C(\Omega)$, i.e., every sequence $\{f_n\}, f_n \in X$, has a convergent subsequence f_{n_j} converging to f_∞ in X in the sup-norm topology.

4. Let $\sum_{n=0}^{\infty} a_n$ be a convergent series with nonnegative terms, and S be its sum. For $f(x) = \sum_{n=0}^{\infty} a_n x^n$ show that $\lim_{x \rightarrow +1^-} f(x) = S$.

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function, and $\lim_{x \rightarrow -\infty} f'(x) = -\infty, \lim_{x \rightarrow +\infty} f'(x) = +\infty$. Show that for any $A \in \mathbb{R}$ there exists $a \in \mathbb{R}$ such that $f'(a) = A$.
Warning: $f'(x)$ may not be continuous.

6. Consider the function

$$K(t, x) = xt^{-\frac{3}{2}}e^{-\frac{x^2}{4t}}$$

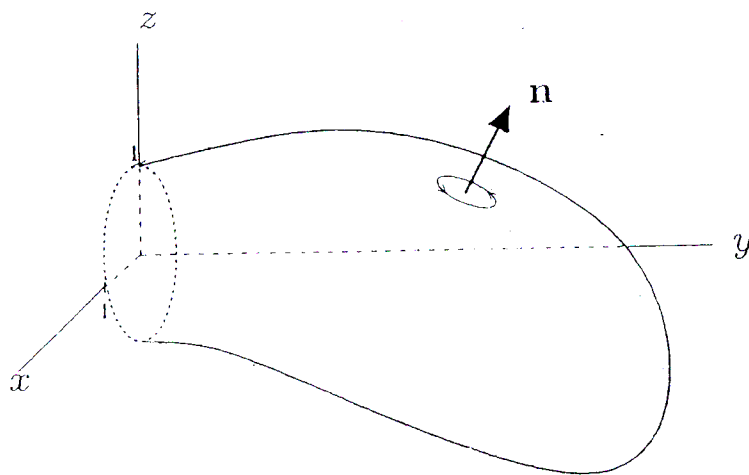
defined for all $x \in \mathbb{R}$ and $t > 0$. Clearly, $K(t, 0) = 0$ for $t > 0$. Show that $K(t, x) \rightarrow 0$ as $t \rightarrow 0^+$ for any fixed x . Can you define K at $(0, 0)$ to make it continuous there?

7. Calculate

$$\iint_D \cos\left(\frac{x+2y}{-x+y}\right) dx dy,$$

where D is the triangular region in \mathbb{R}^2 having vertices $(0, 0)$, $(-2, 4)$, $(-3, 3)$.

8. A soap film bubble blown from a circular hoop describes an undetermined region $\Omega \subset \{(x, y, z) \in \mathbb{R}^3 : y \leq 0\}$ having three-dimensional volume equal to 10. Let S denote that portion of $\partial\Omega$ comprised of the soap film (it does not include the unit disk D in the x, z -plane). Suppose a force field $\mathbf{F} = (z^2, 3y + 5, x^3)$ is applied. Find $\int_S \mathbf{F} \cdot \mathbf{n} dA$, where \mathbf{n} is the outward pointing normal on $\partial\Omega$, and dA is the surface element.



9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $f(x) = 0$ if $|x| \geq 1$.

a) Show that the improper integral

$$g(y) = \int_{-\infty}^{\infty} \frac{f(x)}{\sqrt{|x-y|}} dx$$

converges for all $y \in \mathbb{R}$, and $g(y)$ is continuous.

b) Show that, if additionally f is continuously differentiable, then so is g and

$$g'(y) = \int_{-\infty}^{\infty} \frac{f'(x)}{\sqrt{|x-y|}} dx$$

10. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuously differentiable map and $df_a : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be its differential at $a \in \mathbb{R}^n$. Suppose that df_a is positive at any $a \in \mathbb{R}^n$, in the sense that $\langle df_a(x), x \rangle > 0$ for all $a \in \mathbb{R}^n$, and $x \in \mathbb{R}^n - \{0\}$.

Prove that f is injective

Hint: For $a \in \mathbb{R}^n - \{0\}$ consider $g : \mathbb{R} \rightarrow \mathbb{R}^n$ defined by $g(t) = f(ta)$. Find $g'(t)$ and show that $f(a) \neq f(0)$.

**TIER 1 ANALYSIS EXAM
AUGUST 2007**

- (1) Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by setting

$$f(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that f is differentiable at all points $(x, y) \in \mathbb{R}^2$ except $(0, 0)$. Show that f is not differentiable at $(0, 0)$.

- (2) Given $\lambda \in \mathbb{R}$, define $h_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$h_\lambda(x, y) = -x^4 + x^2 + y^2 + \lambda \cdot \sin(x \cdot y).$$

For which values of λ does h_λ have a local minimum at $(0, 0)$? Justify your answer.

- (3) Let $\gamma \subset \mathbb{R}^2$ be the simple closed curve described in polar coordinates by $r = \cos(2\theta)$ where $\theta \in [-\pi/4, \pi/4]$. Suppose that γ is positively oriented. Compute the line integral

$$\int_{\gamma} 3y \, dx + x \, dy.$$

Provide the details of your computation.

- (4) Let X be a metric space such that $d(x, y) \leq 1$ for every $x, y \in X$, and let $f : X \rightarrow \mathbb{R}$ be a uniformly continuous function. Does it follow that f must be bounded? Justify your answer with either a proof or a counterexample.

- (5) Let

$$f(x, y) = (x + e^{2y} - 1, \sin(x^2 + y)),$$

and let

$$h(x, y) = (1 + x)^5 - e^{4y}.$$

Show that there exists a continuously differentiable function $g(x, y)$ defined in a neighborhood of $(0, 0)$ such that $g(0, 0) = 0$ and $g \circ f = h$. Compute $\frac{\partial g}{\partial y}(0, 0)$.

- (6) Let c_1, c_2, \dots be an infinite sequence of distinct points in the interval $[0, 1]$. Define $f : [0, 1] \rightarrow \mathbb{R}$ by setting $f(x) = 1/n$ if $x = c_n$ and $f(x) = 0$ if $x \notin \{c_n\}$. State the definition of a Riemann integrable function, and directly use this definition to show that

$$\int_0^1 f(x) dx$$

exists.

- (7) Show that the formula

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{\int_0^x t \sin(\frac{t}{n}) dt}$$

defines a function $g : \mathbb{R} \rightarrow \mathbb{R}$. Prove that g is continuously differentiable.

- (8) Consider an unbounded sequence $0 < a_1 < a_2 < \dots$, and set

$$s = \limsup_{n \rightarrow \infty} \frac{\log n}{\log a_n}.$$

Show that the series

$$\sum_{n=1}^{\infty} a_n^{-t}$$

converges for $t > s$ and diverges for $t < s$.

- (9) Define a sequence $\{a_n\}$ by setting $a_1 = 1/2$ and $a_{n+1} = \sqrt{1 - a_n}$ for $n \geq 2$. Does the sequence a_n converge? If so, what is the limit? Justify your answer with a proof.

Tier 1 Analysis Exam

January 2008

1. Give an example of a function $f : [0, \infty) \rightarrow \mathbb{R}$ that satisfies the three conditions:

- (i) $f(x) \geq 0$ for all $x \geq 0$,
- (ii) for every $M > 0$, $\sup_{x > M} f(x) = \infty$,
- (iii) $\int_0^\infty f(x) dx < \infty$,

or else prove that no such function exists.

2. Determine whether the series

$$\sum_{n=1}^{\infty} \ln \left(n \sin \frac{1}{n} \right)$$

is convergent (conditionally or absolutely) or divergent.

3. Let S be a closed, nonempty subset of \mathbb{R}^n that is convex in the sense that if q_1 and q_2 are any two points in S , then $\lambda q_1 + (1 - \lambda)q_2 \in S$ for all $\lambda \in (0, 1)$. Given any $p \in \mathbb{R}^n \setminus S$, let

$$m = \inf_{q \in S} \{\|p - q\|\}$$

where $\|\cdot\|$ denotes the usual Euclidean norm. Prove that there exists exactly one point $q \in S$ that achieves this infimum.

4. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \frac{xy^2}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Notice that f is C^1 on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

(i) Show that f is continuous at $(0, 0)$.

(ii) Show that all the directional derivatives of f at $(0, 0)$ exist by calculating the directional derivative of f at $(0, 0)$ in the direction V , for any given unit vector $V = (\cos \theta, \sin \theta)$. (Recall that the directional derivative of f at a point p in direction V is by definition, $\frac{d}{dt} \Big|_{t=0} f(p + tV)$.)

(iii) Show that f is not differentiable at $(0, 0)$.

5. Let $f = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be continuously differentiable, and assume that the 2×2 matrix $Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_j}(x) \end{pmatrix}$ is invertible for all $x \in \mathbb{R}^2$. Assume moreover that, for any

compact set $K \subset \mathbb{R}^2$, $f^{-1}(K)$ is compact. Prove that f is onto.

6. Let f be a continuous function on $[0, \infty)$ such that $0 \leq f(x) \leq Cx^{-1-\rho}$ for all $x > 0$, and for some constants $C, \rho > 0$. Let $f_k(x) = kf(kx)$.

(i) Show that $\lim_{k \rightarrow \infty} f_k(x) = 0$ for any $x > 0$ and that the convergence is uniform on $[r, \infty)$ for any $r > 0$.

(ii) Show that f_k does not converge to zero uniformly on $(0, \infty)$, unless f is identically 0.

7. Let f and f_k be defined as in the previous problem.

(i) Show that the limit $\lim_{k \rightarrow \infty} \int_0^1 f_k(x) dx$ exists.

(ii) Denote by a the limit in (i). Show that $\lim_{k \rightarrow \infty} \int_0^1 f_k(x)g(x) dx = ag(0)$ for any Riemann integrable function g on $[0, 1]$ that is continuous at 0.

(Note: The result of the previous problem is not necessarily needed for solving this problem.)

8. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$ for all $x \in (0, 1)$. Show that, for any positive integer n

$$\left| \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k-1}{n}\right) \right| \leq \frac{M}{n} .$$

9. Consider the quartic equation with real coefficients

$$x^4 + a_0x^3 + a_1x^2 + 2a_2x + a_3 = 0 .$$

Show that there exists $\delta > 0$ such that if $|a_i - 1| < \delta$, $i = 0, 1, 2, 3$, then the equation above has a real solution which depends smoothly on the a_i 's.

10. Compute the line integral

$$\int_C \frac{x dy - y dx}{x^2 + y^2} ,$$

where C is a simple closed C^1 curve around the origin of the xy -plane, and oriented counterclockwise.

TIER I ANALYSIS EXAM

August 2008

Do all 10 problems; they all count equally.

Problem 1. Suppose that I_1, \dots, I_n are disjoint closed subintervals of \mathbb{R} . If f is uniformly continuous on each of the intervals, prove that f is uniformly continuous on $\bigcup_{j=1}^n I_j$.

Does this still hold if the intervals are open?

Problem 2. Suppose that f is a continuous function from $[0, 1]$ into \mathbb{R} and that $\int_0^1 f(x) dx = 0$.

Prove that there is at least one point, x_0 , in $[0, 1]$, where $f(x_0) = 0$.

Does this still hold if f is Riemann integrable but not continuous?

Problem 3. Suppose that f is a continuous function from $[a, b]$ into \mathbb{R} which has the property that, for any point $x \in [a, b]$, there is another point $x' \in [a, b]$ such that $|f(x')| \leq |f(x)|/2$.

Prove that there exists a point $x_0 \in [a, b]$ where f vanishes, that is, $f(x_0) = 0$.

Problem 4. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$f(x, y) = (\sin(y) - x, e^x - y), \quad g(x, y) = (xy, x^2 + y^2).$$

Compute $(g \circ f)'(0, 0)$.

Problem 5. Prove that there exists a positive number θ_0 such that the following holds: For each $\theta \in [0, \theta_0]$, there exist real numbers x and y (with $xy > -1$) such that

$$2x + y + e^{xy} = \cos(\theta^3), \quad \text{and} \quad \log(1 + xy) + \sin(x + y^2) = \sqrt{\theta}.$$

(*Hint:* First evaluate the left side of each of these two equations for $x = y = 0$.)

Problem 6. If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent series of real numbers it is well-known that their *Cauchy product series* $\sum_{n=0}^{\infty} c_n$ also converges, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0, \quad n = 0, 1, \dots$$

Show that this assertion is no longer true if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are merely conditionally convergent.

Problem 7. (a.) Let C be the line segment joining the points (x_1, y_1) and (x_2, y_2) in \mathbb{R}^2 .

Prove that $\int_C x dy - y dx = x_1 y_2 - x_2 y_1$.

(b.) Suppose further that $(x_1, y_1), \dots, (x_n, y_n)$ are vertices of a polygon in \mathbb{R}^2 , in counterclockwise order.

Prove that the area of the polygon is equal to

$$\frac{1}{2} [(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_n y_1 - x_1 y_n)].$$

Problem 8. Prove that there exist a positive integer n and real numbers a_0, a_1, \dots, a_n such that

$$\left| \left(\sum_{k=0}^n \frac{a_k}{x^k} \right) - \exp \left(\frac{\sin(e^x)}{\sqrt{x}} \right) \right| \leq 10^{-6} \quad \text{for all } x \in [1, \infty).$$

Problem 9. Prove that the series $\sum_{n=1}^{\infty} n^{-x}$ can be differentiated term by term on its interval of convergence.

Problem 10. Suppose that, for each positive integer n ,

$$f_n : [0, 1] \rightarrow \mathbb{R}$$

is a continuous function that satisfies $f_n(0) = 0$ and has a continuous derivative f'_n on $(0, 1)$ such that $|f'_n(x)| \leq 9000$ for all $x \in (0, 1)$.

Prove that there exists a subsequence $f_{n_1}, f_{n_2}, f_{n_3}, \dots$ such that the following holds:

For every Riemann integrable function $g : [0, 1] \rightarrow \mathbb{R}$, there exists a real number L (which may depend on the function g) such that

$$\lim_{k \rightarrow \infty} \int_0^1 g(x) f_{n_k}(x) dx = L.$$

(*Note.* You may take for granted and freely use standard basic facts about Riemann integrals, including, e.g. the fact that a Riemann integrable function is bounded, and that linear combinations, products, and absolute values of Riemann integrable functions are Riemann integrable.)

Tier I Analysis Exam

January 2009

Try to work all questions. They all are worth the same amount.

1. Assume f and g are uniformly continuous functions from $\mathbb{R}^1 \rightarrow \mathbb{R}^1$. If both f and g are also bounded, show that fg is also uniformly continuous. Then give an example to show that in general, if f and g are both uniformly continuous but not both bounded, then the product is not necessarily uniformly continuous. (Verify clearly that your counter-example is not uniformly continuous.)
2. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are \mathcal{C}^2 functions, $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a \mathcal{C}^1 function and assume

$$f(0) = g(0) = 0, \quad f'(0) = g'(0) = h(0,0) = 1.$$

Show that the function $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$H(x, y) := \int_0^{f(x)} \int_0^{g(y)} h(s, t) ds dt + \frac{1}{2}x^2 + by^2$$

has a local minimum at the origin provided that $b > \frac{1}{2}$ while it has a saddle at the origin if $b < \frac{1}{2}$.

3. Let $H = \{(x, y, z) \mid z > 0 \text{ and } x^2 + y^2 + z^2 = R^2\}$, i.e. the upper hemisphere of the sphere of radius R centered at 0 in \mathbb{R}^3 . Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the vector field

$$F(x, y, z) = \left\{ x^2(y^2 - z^3), xzy^4 + e^{-x^2}y^4 + y, x^2y(y^2x^3 + 3)z + e^{-x^2-y^2} \right\}$$

Find $\int_H F \cdot \hat{n} dS$ where \hat{n} is the outward (upward) pointing unit surface normal and dS is the area element.

4. Let D be the square with vertices $(2,2)$, $(3,3)$, $(2,4)$, $(1,3)$. Calculate the improper integral

$$\int \int_D \ln(y^2 - x^2) dx dy .$$

5. Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ is a \mathcal{C}^4 function with the property that at some point $(x_0, y_0) \in \mathbb{R}^2$ all of the first and second order partial derivatives of f vanish. Suppose also that at least one partial derivative of third order does not vanish at (x_0, y_0) . Prove that f can have neither a local maximum nor a local minimum at this critical point.

6. Prove that the series $\sum_{n=1}^{\infty} \frac{nx}{1 + n^2 \log^2(n)x^2}$ converges uniformly on $[\varepsilon, \infty)$ for any $\varepsilon > 0$.

7. Suppose that $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is of class \mathcal{C}^1 , that $f(0, 0, 0) = 0$, and

$$f_2(0, 0, 0) \neq 0, \quad f_3(0, 0, 0) \neq 0, \quad \text{and} \quad f_2(0, 0, 0) + f_3(0, 0, 0) \neq -1$$

where $f_k = \frac{\partial f}{\partial x_k}$. Show that the system

$$f(x_1, f(x_1, x_2, x_3), x_3) = 0$$

$$f(x_1, x_2, f(x_1, x_2, x_3)) = 0$$

defines \mathcal{C}^1 functions $x_2 = \varphi(x_1)$, and $x_3 = \psi(x_1)$ for x_1 in a neighborhood of 0 satisfying

$$f(x_1, f(x_1, \varphi(x_1), \psi(x_1)), \psi(x_1)) = 0$$

$$f(x_1, \varphi(x_1), f(x_1, \varphi(x_1), \psi(x_1))) = 0.$$

8. For each $b \in [1, e]$, consider the sequence of real numbers governed by the recurrence relation

$$a_{n+1} = \left(\sqrt[n]{b}\right)^{a_n} \quad \text{for } n = 0, 1, 2, \dots \quad \text{with } a_0 = \sqrt[b]{b} \quad \text{i.e.} \quad \{\sqrt[b]{b}, \sqrt[n]{b}^{\sqrt[b]{b}}, \sqrt[n]{b}^{\sqrt[n]{b}^{\sqrt[b]{b}}}, \sqrt[n]{b}^{\sqrt[n]{b}^{\sqrt[n]{b}^{\sqrt[b]{b}}}}, \dots\}.$$

Show that this sequence converges and find the limit.

9. For each positive integer n , define $x_n : [-1, 1] \rightarrow \mathbb{R}$ by

$$x_n(t) = \begin{cases} -1 & \text{if } -1 \leq t \leq -1/n \\ nt & \text{if } -1/n < t < 1/n \\ 1 & \text{if } 1/n \leq t \leq 1 \end{cases}$$

(a) Show that $\{x_n\}$ is a Cauchy sequence in the metric space $(\mathcal{C}([-1, 1]), d)$, where $\mathcal{C}([-1, 1])$ denotes the set of continuous functions defined on $[-1, 1]$ and d denotes the metric given by

$$d(x, y) = \int_{-1}^1 |x(t) - y(t)| dt .$$

(b) Show that $(\mathcal{C}([-1, 1]), d)$ is not complete.

Tier 1 Analysis Exam

August, 2009

Show all work, and justify all answers.

This exam has 9 problems.

\mathbf{R} will denote the real numbers, and $\|\cdot\|$ will denote the usual Euclidean norm.

1. Define the statement: “ $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ is differentiable at $(0,0)$,” and show that the function $f(x,y) = x|y|^{\frac{1}{2}}$ is differentiable at $(0,0)$.

2. Show that the series

$$2 \sin \frac{1}{3x} + 4 \sin \frac{1}{9x} + \cdots + 2^n \sin \frac{1}{3^n x} + \cdots$$

converges absolutely for $x \neq 0$ but does not converge uniformly on any interval $(0, \epsilon)$ with $\epsilon > 0$.

3. Let $V(n,r)$ be the volume of the ball $\{x \in \mathbf{R}^n : \|x\| \leq r\}$.

(a) Show that $V(n,r) = c_n r^n$ for some constant c_n depending only on n .

(b) Find $\lim_{n \rightarrow \infty} c_n$.

4. Suppose that $x \neq 0$. Show that

$$\lim_{n \rightarrow \infty} \frac{1 + \cos(x/n) + \cos(2x/n) + \cdots + \cos((n-1)x/n)}{n} = \frac{\sin(x)}{x}$$

5. Let $X = \{x = (x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1^2 + x_2^2 + x_3^2 - x_4^2 = 2, \text{ and } x_1 + x_2 + x_3 + x_4 = 2\}$. For which points $p \in X$ is it possible to find a product of open intervals $V = I_1 \times I_2 \times I_3 \times I_4$ containing p such that $X \cap V$ is the graph of a function expressing some of the variables x_1, x_2, x_3, x_4 in terms of the others? If there are any points in X where this is not possible, explain why not.

6. Let a and b be two points of \mathbf{R}^2 . Let $\sigma_n : [0, 1] \rightarrow \mathbf{R}^2$ be a sequence of continuously differentiable constant speed curves with $\|\sigma_n'(t)\| = L_n$ for all $t \in [0, 1]$ and $\sigma_n(0) = a$ and $\sigma_n(1) = b$ for all n . Suppose that $\lim_{n \rightarrow \infty} L_n = \|b - a\|$. Show that σ_n converges uniformly to σ , where $\sigma(t) = a + t(b - a)$ for $t \in [0, 1]$.

7. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a function; and let its n -th derivative, denoted $f^{(n)}$, exist for all n . Suppose that the sequence $f^{(n)}$, $n = 1, 2, 3, \dots$ converges uniformly on compact subsets to a function g . Show that there is a constant c such that $g(x) = ce^x$.

8. Let $M = \{(x, y, z) \in \mathbf{R}^3 : y = 9 - x^2, y \geq 0, \text{ and } 0 \leq z \leq 1\}$. Orient M so that the unit normal \vec{n} is in the positive y -direction along the line $x = 0, y = 3$. Let \vec{F} be the vector field on \mathbf{R}^3 given by $\vec{F} = (2x^3yz, y + 3x^2y^2z, -6x^2yz^2)$.

(a) What is $\operatorname{div} \vec{F}$?

(b) Use the Divergence Theorem to express the flux of \vec{F} across M (that is, $\int_M \vec{F} \cdot \vec{n} dS$, where dS is the surface area element) in terms of some other (easier) integrals.

(c) Calculate $\int_M \vec{F} \cdot \vec{n} dS$ by evaluating the integrals in part (b).

9. Let (X, d) be a compact metric space. Suppose that $h : X \rightarrow Y \subset X$ is a map which preserves d , or in other words, $d(h(x_1), h(x_2)) = d(x_1, x_2)$ for all $x_1, x_2 \in X$. Show that $Y = X$.

Department of Mathematics—Tier 1 Analysis Examination

January 7, 2010

Notation: In problems 2, 3, and 9 the notation ∇f denotes the n -tuple of first-order partial derivatives of a function f mapping an open set in \mathbf{R}^n into \mathbf{R} .

1. Let E be a closed and bounded set in \mathbf{R}^n and let $f : E \rightarrow \mathbf{R}$. Suppose that for each $x \in E$ there are positive numbers r and M depending on x such that $f(y) \geq -M$ for all $y \in E$ satisfying $|y - x| < r$. Prove that there is a positive number \bar{M} such that $f(y) \geq -\bar{M}$ for all $y \in E$.
2. Let V be a convex open set in \mathbf{R}^2 and let $f : V \rightarrow \mathbf{R}$ be continuously differentiable in V . Show that if there is a positive number M such that $|\nabla f(x)| \leq M$ for all $x \in V$, then there is a positive number L such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in V$.

Is this result still true if V is instead assumed to be open and connected? Prove or disprove with a counterexample.

3. Let f be a C^2 mapping of a neighborhood of a point $x_0 \in \mathbf{R}^n$ into \mathbf{R} . Assume that x_0 is a critical point of f and that the second derivative matrix $f''(x_0)$ is positive definite. Prove that there is a neighborhood V of x_0 such that zero is an interior point of the set $\{\nabla f(y) : y \in V\}$.
4. Suppose that F and G are differentiable maps of a neighborhood V of a point $x_0 \in \mathbf{R}^n$ into \mathbf{R} and that $F(x_0) = G(x_0)$. Next let $f : V \rightarrow \mathbf{R}$ and suppose that $F(x) \leq f(x) \leq G(x)$ for all $x \in V$. Prove that f is differentiable at $x = x_0$.
5. Let $\{g_k\}_{k=1}^\infty$ be a sequence of continuous real-valued functions on $[0, 1]$. Assume that there is a number M such that $|g_k(x)| \leq M$ for every k and every $x \in [0, 1]$ and also that there is a continuous real-valued function g on $[0, 1]$ such that

$$\int_0^1 g_k(x)p(x)dx \rightarrow \int_0^1 g(x)p(x)dx \quad \text{as } k \rightarrow \infty$$

for every polynomial p . Prove that $|g(x)| \leq M$ for every $x \in [0, 1]$ and that

$$\int_0^1 g_k(x)f(x)dx \rightarrow \int_0^1 g(x)f(x)dx$$

for every continuous f .

6. Let $\{a_k\}$ be a sequence of positive numbers converging to a positive number a . Prove that $(a_1 a_2 \cdots a_k)^{1/k}$ also converges to a .
7. Compute rigorously $\lim_{n \rightarrow \infty} \left[\frac{1}{n + \sqrt{n}} \sum_{k=1}^n \sin\left(\frac{k}{n}\right) \right]$.
8. Let $\{a_k\}_{k=1}^\infty$ be a sequence of numbers satisfying $|a_k| \leq k^2/2^k$ for all k and let $f : [0, 1] \times \mathbf{R} \rightarrow \mathbf{R}$ be continuous. Prove that the following limit exists:

$$\lim_{n \rightarrow \infty} \int_0^1 f\left(x, \sum_{k=1}^n a_k x^k\right) dx .$$

9. Let $g : \mathbf{R}^2 \rightarrow (0, \infty)$ be C^2 and define $\Sigma \subset \mathbf{R}^3$ by $\Sigma = \{(x_1, x_2, g(x_1, x_2)) : x_1^2 + x_2^2 \leq 1\}$. Assume that Σ is contained in the ball B of radius R centered at the origin in \mathbf{R}^3 and that each ray through the origin intersects Σ at most once. Let E be the set of points $x \in \partial B$ such that the ray joining the origin to x intersects Σ exactly once. Derive an equation relating the area of E , R , and the integral

$$\int_{\Sigma} \nabla \Gamma(x) \cdot N(x) dS$$

where $\Gamma(x) = 1/|x|$, $N(x)$ is a unit normal vector on Σ , and dS represents surface area.

Tier I Analysis Exam
August, 2010

- **Be sure to fully justify all answers.**
 - **Scoring:** Each one of the 10 problems is worth 10 points.
 - **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
 - Please be sure that you assemble your test with the problems presented in correct order.
-

- (1) Let A and B be bounded sets of positive real numbers and let $AB = \{ab \mid a \in A, b \in B\}$. Prove that $\sup AB = (\sup A)(\sup B)$.
- (2) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called *proper* if $f^{-1}(C)$ is compact for every compact set C . Prove or give a counterexample: if f and g are continuous and proper, then the product fg is proper.
- (3) (a) Prove or give a counterexample: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function and $f(x) > x^2$ for all x , then given any $M \in \mathbb{R}$ there is an x_0 such that $|f'(x_0)| > M$.
- (b) Prove or give a counterexample: If $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a differentiable function and $\|f(x, y)\| > \|(x, y)\|^2$ for all (x, y) , then given any $M \in \mathbb{R}$ there is an $(x_0, y_0) \in \mathbb{R}^2$ such that $|\det(Df(x_0, y_0))| > M$.
- (4) Suppose that $\{f_n\}$ is a sequence of continuous functions defined on the interval $[0, 1]$ converging *uniformly* to a function f_0 . Let $\{x_n\}$ be a sequence of points converging to a point x_0 with the property that for each n , $f_n(x_n) \geq f_n(x)$ for all $x \in [0, 1]$. Prove that $f_0(x_0) \geq f_0(x)$ for all $x \in [0, 1]$.

- (5) Let f be continuous at $x = 0$, and assume

$$\lim_{x \rightarrow 0} \frac{f(2x) - f(x)}{x} = L.$$

Prove that $f'(0)$ exists and $f'(0) = L$.

- (6) Let $R = \{(x, y) \mid 0 \leq x, 5|y| \leq 3|x|, x^2 - y^2 \leq 1\}$, a compact region in \mathbb{R}^2 . For some region $S \subset \mathbb{R}^2$, the function $F: S \rightarrow R$ given by $F(r, \theta) = (r \cosh \theta, r \sinh \theta)$ is one-to-one and onto. Determine S and use this change of variable to compute the integral

$$\iint_R \frac{dx dy}{1 + x^2 - y^2}.$$

(Recall that $\cosh \theta = \frac{e^\theta + e^{-\theta}}{2}$ and $\sinh \theta = \frac{e^\theta - e^{-\theta}}{2}$.)

(7) Let $d(x) = \min_{n \in \mathbb{Z}} |x - n|$, where \mathbb{Z} is the set of all integers.

(a) Prove that $f(x) = \sum_{n=0}^{\infty} \frac{d(10^n x)}{10^n}$ is a continuous function on \mathbb{R} .

(b) Compute explicitly the value of $\int_0^1 f(x) dx$.

(8) Suppose f and φ are continuous real valued functions on \mathbb{R} . Suppose $\varphi(x) = 0$ whenever $|x| > 5$, and suppose that $\int_{\mathbb{R}} \varphi(x) dx = 1$. Show that

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{\mathbb{R}} f(x - y) \varphi\left(\frac{y}{h}\right) dy = f(x)$$

for all $x \in \mathbb{R}$.

(9) Let $f(x, y, z)$ and $g(x, y, z)$ be continuously differentiable functions defined on \mathbb{R}^3 . Suppose that $f(0, 0, 0) = g(0, 0, 0) = 0$. Also, assume that the gradients $\nabla f(0, 0, 0)$ and $\nabla g(0, 0, 0)$ are linearly independent. Show that for some $\epsilon > 0$ there is a differentiable curve $\gamma: (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3$ with nonvanishing derivative such that $\gamma(0) = (0, 0, 0)$ and $f(\gamma(t)) = g(\gamma(t)) = 0$ for all $t \in (-\epsilon, \epsilon)$.

(10) Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \text{ and } z = e^{x^2 + 2y^2}\}$. So, S is that part of the surface described by $z = e^{x^2 + 2y^2}$ that lies inside the cylinder $x^2 + y^2 = 1$. Let the path $C = \partial S$. Choose (specify) an orientation for C and compute

$$\int_C (-y^3 + xz) dx + (yz + x^3) dy + z^2 dz .$$

TIER 1 ANALYSIS EXAM, JANUARY 2011

- Solve the following 10 problems, justifying all answers.
- Write the solution of each problem on a separate, clearly identified page.

(1) In this problem we use the notation

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

for the Euclidean norm of a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Let $f : [0, 1] \rightarrow \mathbb{R}^n$ be a continuous function. Show that

$$\left| \int_0^1 f(t) dt \right| \leq \int_0^1 |f(t)| dt.$$

(2) A sequence

$$A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}, \quad n \geq 1,$$

of real matrices is said to converge if the sequences $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty, (c_n)_{n=1}^\infty, (d_n)_{n=1}^\infty$ converge. Fix a real matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and define

$$A_n = A - \frac{1}{3!}A^3 + \cdots + \frac{(-1)^n}{(2n+1)!}A^{2n+1}, \quad n \geq 1.$$

Show that the sequence $(A_n)_{n=1}^\infty$ converges. (The limit is denoted $\sin(A)$.)

(3) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with the following property: for every positive integer n , and every $x, y \in \mathbb{R}$ such that $|x| + |y| > n^2$ and $|x - y| < 1/n^2$, we have $|f(x) - f(y)| < 1/n$. Show that f is uniformly continuous.

(4) Determine the area enclosed by the curve

$$c(t) = (3 \cos t - \cos(3t), 3 \sin t - \sin(3t)), \quad t \in [0, 2\pi].$$

You can take it for granted that this is a simple curve.

(5) Determine the volume of the solid $\{(x, y, z) : \sqrt{x} + \sqrt{y} + \sqrt{z} \leq 1, x, y, z \geq 0\}$.

(6) Consider a differentiable, strictly decreasing function $f : [0, 1] \rightarrow [0, 1]$, and let $a \in [0, 1]$ satisfy $f(a) = a$. (There obviously exists exactly one such point.) Assume that $f'(a) < -1$. Define a sequence $(x_n)_{n=0}^\infty$ by setting $x_0 = 0$ and $x_{n+1} = f(x_n)$ for $n \geq 0$. Show that the sequence $(x_n)_{n=0}^\infty$ does not converge.

(7) Show that there exists a differentiable function $f(x)$ defined in a neighborhood of $x_0 = \sqrt{2}$ such that $x^{f(x)} = f(x)$.

(8) Given a sequence $f_n : [0, 1] \rightarrow [0, 1]$ of continuous functions, define $g_n : [0, 1] \rightarrow \mathbb{R}$ by setting

$$g_n(x) = \int_0^1 \frac{f_n(t)}{(t-x)^{1/3}} dt, \quad x \in [0, 1].$$

(Observe that this is an improper Riemann integral.) Show that the sequence $(g_n)_{n \in \mathbb{N}}$ has a uniformly convergent subsequence.

- (9) A sequence $(a_n)_{n=1}^{\infty}$ satisfies the inequality $|\sum_{k=1}^n a_k| \leq \sqrt{n}$ for all $n \geq 1$. Show that the series

$$\sum_{k=1}^{\infty} \frac{a_k}{k}$$

converges.

- (10) Show that there does not exist a sequence $I_n = [a_n, b_n], n = 1, 2, \dots$ of nonempty, pairwise disjoint intervals such that $\bigcup_n I_n = [0, 1]$.

Tier 1 Analysis Exam: August 2011

Do all nine problems. They all count equally. Show all computations.

1. Let (X, d) be a compact metric space. Let $f : X \rightarrow X$ be continuous. Fix a point $x_0 \in X$, and assume that $d(f(x), x_0) \geq 1$ whenever $x \in X$ is such that $d(x, x_0) = 1$. Prove that $U \setminus f(U)$ is an open set in X , where $U = \{x \in X : d(x, x_0) < 1\}$.

2. Let $f_1 : [a, b] \rightarrow \mathbb{R}$ be a Riemann integrable function. Define the sequence of functions $f_n : [a, b] \rightarrow \mathbb{R}$ by

$$f_{n+1}(x) = \int_a^x f_n(t) dt,$$

for each $n \geq 1$ and each $x \in [a, b]$. Prove that the sequence of functions

$$g_n(x) = \sum_{m=1}^n f_m(x)$$

converges uniformly on $[a, b]$.

3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be differentiable everywhere. Assume $f(-\sqrt{2}, -\sqrt{2}) = 0$, and also that

$$\left| \frac{\partial f}{\partial x}(x, y) \right| \leq |\sin(x^2 + y^2)|$$

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq |\cos(x^2 + y^2)|$$

for each $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Prove that

$$|f(\sqrt{2}, \sqrt{2})| \leq 4.$$

4. Let q_1, q_2, \dots be an indexing of the rational numbers in the interval $(0, 1)$. Define the function $f(x) : (0, 1) \rightarrow (0, 1)$, by

$$f(x) = \sum_{j:q_j < x} 2^{-j}.$$

(Here the sum is over all positive integers j such that $q_j < x$.)

- Show that f is discontinuous at every rational number in $(0, 1)$.
- Show that f is continuous at every irrational number in $(0, 1)$.

5. Show that the map $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$\Phi(\theta, \phi) = (\sin \phi \cdot \cos \theta, \sin \phi \cdot \sin \theta),$$

is invertible in a neighborhood of $(\theta_0, \phi_0) = (\frac{\pi}{6}, \frac{\pi}{4})$ and find the partial derivatives of the inverse at the point $(\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4})$.

6. Let A be a domain in \mathbb{R}^2 whose boundary γ is a smooth, positively oriented curve.

a. Find a particular pair of functions $P : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $Q : \mathbb{R}^2 \rightarrow \mathbb{R}$ so that $\int_{\gamma} Pdx + Qdy$ equals the area of the domain A .

b. Let $|A|$ be the area of A . Find a function $R : \mathbb{R}^2 \rightarrow \mathbb{R}$ so that

$$\frac{1}{|A|} \int_{\gamma} Rdx + Rdy,$$

equals the average value of the square of the distance from the origin to a point of A .

7. Let C be a smooth simple closed curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C zdx - 2xdy + 3ydz$$

depends only on the orientation of C and on the area of the region enclosed by C but not on the shape of C or its location in the plane.

8. For each $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ define $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$. Consider

$$F(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^\lambda}, \quad \mathbf{x} \neq 0, \lambda > 0.$$

(i) Is there a value of λ for which F is divergence free?

(ii) Let $E : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by

$$E(\mathbf{y}) = q \frac{\mathbf{y}}{|\mathbf{y}|^3}$$

where q is a positive real number. Let $S(\mathbf{x}, a)$ denote the sphere of radius $a > 0$ centered at \mathbf{x} . Assume $|\mathbf{x}| \neq a$. Compute

$$\int_{S(\mathbf{x}, a)} E \cdot n \, dA$$

where dA is the surface area element and n is the unit outward normal on $S(\mathbf{x}, a)$.

9. Let $x_1 \in \mathbb{R}$. Define the sequence $(x_n)_{n \geq 2}$ by

$$x_{n+1} = x_n + \frac{\sqrt{|x_n|}}{n^2},$$

for each $n \geq 1$. Show that x_n is convergent.

Tier I Analysis

January 3, 2012

Solve all 10 problems, justifying all answers.

1. For $(x, y) \in \mathbb{R}^2$, let

$$f(x, y) = \begin{cases} [(2x^2 - y)(y - x^2)]^{1/4}, & \text{for } x^2 \leq y \leq 2x^2; \\ 0, & \text{otherwise.} \end{cases}$$

Show that all directional derivatives of f exist at $(0, 0)$, but f is not differentiable at $(0, 0)$.

2. Let $(a_n)_{n=1}^{\infty}$ be a monotonically decreasing sequence of positive real numbers and assume $\sum_{n=1}^{\infty} a_n < \infty$. Show that $\lim_{n \rightarrow \infty} na_n = 0$.
3. For $(x, y) \in \mathbb{R}^2$, let $f(x, y) = 5x^2 + xy^3 - 3x^2y$. Find the critical points for f , and for each critical point determine whether it is a local maximum, local minimum or a saddle point.
4. Establish the convergence or divergence of the improper integral

$$\int_0^{\infty} \sin(x^2) dx.$$

5. Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be sequences of functions from \mathbb{R} to \mathbb{R} . Assume that

- (a) the partial sums $F_n = \sum_{k=1}^n f_k$ are uniformly bounded,
(b) $g_n \rightarrow 0$ uniformly,
(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \dots$, for all $x \in \mathbb{R}$.

Prove that $\sum_{n=1}^{\infty} f_n g_n$ converges uniformly. *Hint:* Use the fact that

$$\sum_p^q f_n g_n = \sum_p^{q-1} F_n (g_n - g_{n+1}) + F_q g_q - F_{p-1} g_p.$$

(If you make use of this fact, you are required to prove it.)

6. Let

$$X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^4 + x_2^4 + x_3^4 + x_4^4 = 64 \text{ and } x_1 + x_2 + x_3 + x_4 = 8\}.$$

For which points $p \in X$ is it possible to find a product of open intervals $V = I_1 \times I_2 \times I_3 \times I_4$ containing p such that $X \cap V$ is the graph of a function expressing two of the variables x_1, x_2, x_3, x_4 in terms of the other two? If there are any points in X where this is not possible, explain why not.

7. Let $\mathbf{F} : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2$ be given by

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

and suppose for $j = 1, 2$ we have one-to-one C^1 maps $\gamma_j : [0, 1] \rightarrow \mathbb{R}^2$, such that $\gamma_j(0) = p$ and $\gamma_j(1) = q$ for some $p, q \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Assume furthermore that $\gamma_j(t) \neq (0, 0)$ and $\gamma_j'(t) \neq 0$ for all $t \in [0, 1]$, and $\gamma_1((0, 1)) \cap \gamma_2((0, 1)) = \emptyset$. Carefully demonstrate that

$$\int_{\Gamma_1} \mathbf{F} \cdot \mathbf{T}_1 \, ds = \int_{\Gamma_2} \mathbf{F} \cdot \mathbf{T}_2 \, ds + 2\pi k, \text{ for either } k = 0, 1 \text{ or } -1,$$

where $\Gamma_j := \gamma_j([0, 1])$, \mathbf{T}_j denotes the unit tangent vector to γ_j and s is the arc length parameter.

8. Suppose $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is any C^1 function, and let $g : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ be given by $g(x, y) := \ln(\sqrt{x^2 + y^2})$. Prove that

$$\lim_{\epsilon \rightarrow 0} \int_{\partial B_\epsilon} (\phi \nabla g \cdot \mathbf{n} - g \nabla \phi \cdot \mathbf{n}) \, ds = 2\pi \phi(0, 0),$$

where B_ϵ denotes the disk centered at $(0, 0)$ of radius ϵ and \mathbf{n} denotes the outer unit normal to the circle ∂B_ϵ .

9. Let $\alpha \in (0, 1]$. A function $f : [0, 1] \rightarrow \mathbb{R}$ is defined to be α -Hölder continuous if

$$N_\alpha(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in [0, 1], x \neq y \right\} < \infty.$$

- (a) Suppose $(f_n)_{n=1}^\infty$ is a sequence of functions from $[0, 1]$ to \mathbb{R} such that for all $n = 1, 2, \dots$ we have $N_\alpha(f_n) \leq 1$ and $|f_n(x)| \leq 1$ for all $x \in [0, 1]$. Show that $(f_n)_{n=1}^\infty$ has a uniformly convergent subsequence.
- (b) Show that (a) is false if the condition “ $N_\alpha(f_n) \leq 1$ ” is replaced by “ $N_\alpha(f_n) < \infty$.”

10. Assume $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function such that

- (a) there exist points x_0 and $x_1 \in \mathbb{R}^n$ with $f(x_0) = 0$ and $f(x_1) = 3$,
- (b) there exist positive constants C_1 and C_2 such that $f(x) \geq C_1|x| - C_2$ for all $x \in \mathbb{R}^n$.

Let $S := \{x \in \mathbb{R}^n : f(x) < 2\}$ and let $K := \{x \in \mathbb{R}^n : f(x) \leq 1\}$. Define the distance from K to ∂S (the boundary of S) by the formula

$$\text{dist}(K, \partial S) := \inf_{p \in K, q \in \partial S} |p - q|.$$

Prove that $\text{dist}(K, \partial S) > 0$. Then give an example of a continuous function f satisfying (a), but $\text{dist}(K, \partial S) = 0$.

August 2012 Tier 1 Analysis Exam

- Be sure to fully justify all answers.
- Scoring: Each one of the 10 problems is worth 10 points.
- Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write the problem number on each sheet of paper.
- Please be sure that you assemble your test with the problems presented in the correct order.

1. Let

$$f_n(x) = \sum_{k=1}^n (x^k - x^{2k}).$$

- (a) Show that f_n converges pointwise to a function f on $[0, 1]$.
(b) Show that f_n does not converge uniformly to f on $[0, 1]$.

2. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x, y) = \frac{y^3 - \sin^3 x}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

- (a) Compute the directional derivative of f at $(0, 0)$ for an arbitrary direction (u, v) .
(b) Determine whether f is differentiable at $(0, 0)$ and prove your answer.

3. Let E be a nonempty subset of a metric space and let $f : E \rightarrow \mathbb{R}$ be uniformly continuous on E . Prove that f has a unique continuous extension to the closure of E . That is, there exists a unique continuous function $g : \overline{E} \rightarrow \mathbb{R}$ such that $g(x) = f(x)$ for $x \in E$.

4. Let B_r denote the ball $B_r = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < r\}$ and let $f : B_1 \rightarrow \mathbb{R}$ be a continuously differentiable function which is zero in the complement of a compact subset of B_1 . Show that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{B_1 \setminus B_\varepsilon} \frac{x_1 f_{x_1} + x_2 f_{x_2}}{|\mathbf{x}|^2} dx_1 dx_2$$

exists and equals $Cf(\mathbf{0})$ for a constant C which you are to determine.

5. Let E be a nonempty subset of a metric space and assume that for every $\varepsilon > 0$ E is contained in the union of finitely many balls of radius ε . Prove that every sequence in E has a subsequence which is Cauchy.

6. For which exponents $r > 0$ is the limit

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n^2} \frac{n^{r-1}}{n^r + k^r}$$

finite? Prove your answer.

7. Let V be a neighborhood of the origin in \mathbb{R}^2 , and $f : V \rightarrow \mathbb{R}$ be continuously differentiable. Assume that $f(0, 0) = 0$ and $f(x, y) \geq -3x + 4y$ for $(x, y) \in V$. Prove that there is a neighborhood U of the origin in \mathbb{R}^2 and a positive number ε such that, if $(x_1, y_1), (x_2, y_2) \in U$ and $f(x_1, y_1) = f(x_2, y_2) = 0$, then

$$|y_2 - y_1| \geq \varepsilon |x_2 - x_1|.$$

8.

(a) Find necessary and sufficient conditions on functions $h, k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that, given any smooth $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of the form $\mathbf{F} = (F_1(y, z), F_2(x, z), 0)$ and whose divergence is zero, there is a smooth $\mathbf{G} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of the form $\mathbf{G} = (G_1, G_2, 0)$ such that $\nabla \times \mathbf{G} = \mathbf{F}$ in \mathbb{R}^3 and $\mathbf{G} = (h, k, 0)$ on $z = 0$. ($\nabla \times \mathbf{G}$ is the curl of the vector field \mathbf{G} .)

(b) Let \mathbf{F} be as in (a) and evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dA$$

where S is the hemisphere

$$\{(x, y, z) : x^2 + y^2 + z^2 = 1, 0 \leq z \leq 1\},$$

\mathbf{N} is the unit normal on S in the positive z -direction, and dA is the surface area element.

9. Let $f = (f^1, \dots, f^n)$ map an open set U in \mathbb{R}^n into \mathbb{R}^n be C^1 and suppose that, for some $\bar{x} \in U$ the matrix $f'(\bar{x})$ is negative definite (an $n \times n$ matrix A is negative definite if $\xi \cdot A\xi < 0$ for all nonzero $\xi \in \mathbb{R}^n$). Show that there is a positive number ε and a neighborhood V of \bar{x} such that, if y_1, \dots, y_n are any n points in V and if A is the $n \times n$ matrix whose i -th row is $\nabla f^i(y_i)$, then $\xi \cdot A\xi \leq -\varepsilon |\xi|^2$ for all $\xi \in \mathbb{R}^n$.

10. Let f be a C^1 mapping of an open set $U \subset \mathbb{R}^n$ into \mathbb{R}^n and suppose that $f(\bar{x}) = 0$ for some $\bar{x} \in U$ and that $f'(\bar{x})$ is negative definite. Show that there is a neighborhood W of \bar{x} and a positive number δ such that, if a sequence $\{x_k\}_{k=0}^{\infty}$ is generated from the recursion

$$x_{k+1} = x_k + \delta f(x_k)$$

with $x_0 \in W$, then each x_k is in W and $x_k \rightarrow \bar{x}$ as $k \rightarrow \infty$. You may use here the result stated in problem 9 without having solved problem 9.

ANALYSIS TIER 1 EXAM

January 2013

Be sure to fully justify all answers. Each of the 10 problems is worth 10 points. Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write the problem number on each sheet of paper. Please be sure that you assemble your test with the problems presented in the correct order. You have 4 hours.

1. Let X be a bounded closed subset of \mathbb{R}^4 . Let $f: X \rightarrow X$ be a homeomorphism. Write f_n for the n th iterate of f if $n > 0$, for the $-n$ th iterate of f^{-1} if $n < 0$, and for the identity map if $n = 0$. Thus, $f_{n+1}(x) = f(f_n(x))$ for all $n \in \mathbb{Z}$. Write $A(x) := \{f_n(x) : n \in \mathbb{Z}\}$ for $x \in X$. Suppose that $A(x)$ is dense in X for all $x \in X$. Show that for each given $x \in X$ and all $\epsilon > 0$, there exists $n > 0$ such that for all $y \in X$, there exists $k \in [0, n]$ such that $\|f_k(y) - x\| < \epsilon$.

2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function that is differentiable at 0 with $f'(0) \neq 0$. Evaluate

$$\lim_{h \rightarrow 0} \frac{f(h^2 + h^3) - f(h)}{f(h) - f(h^2 - h^3)}.$$

3. Determine all real x for which the following series converges:

$$\sum_{k=1}^{\infty} \frac{k^k}{k!} x^k.$$

You may use the fact that

$$\lim_{k \rightarrow \infty} \frac{k!}{\sqrt{2\pi k} (k/e)^k} = 1.$$

4. (a) Prove that for all $a \in \mathbb{R}$,

$$\left| \sum_{n=1}^{\infty} \frac{a}{n^2 + a^2} \right| < \frac{\pi}{2}.$$

(b) Determine the least upper bound of the set of numbers

$$\left\{ \left| \sum_{n=1}^{\infty} \frac{a}{n^2 + a^2} \right| : a \in \mathbb{R} \right\}.$$

5. Let $f(x)$ be continuous in the interval $I := (0, 1)$. Define

$$D_+f(x_0) := \liminf_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Put

$$S := \{x \in I : D_+f(x) < 0\}.$$

Suppose that the set $f(I \setminus S)$ does not contain any non-empty open interval. (Note: this is $f(I \setminus S)$, not $I \setminus S$.) Prove that $f(x)$ is non-increasing on I .

6. Let $f: (0, 1) \rightarrow \mathbb{R}$ be a function satisfying

$$\forall x, y, \theta \in (0, 1) \quad f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y).$$

Prove that f is continuous on $(0, 1)$.

7. Let $f_0: \mathbb{R} \rightarrow \mathbb{R}$ be the periodic function with period 1 defined on one period by

$$f_0(x) := \begin{cases} x & \text{for } 0 \leq x < \frac{1}{2}, \\ 1 - x & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Let

$$f_k(x) := \frac{1}{10^k} f_0(10^k x) \quad \text{for } k \in \mathbb{N}$$

and let $s_k := f_0 + f_1 + \cdots + f_k$.

(a) Prove that the sequence $\{s_k\}$ converges uniformly on \mathbb{R} to a continuous function $s: \mathbb{R} \rightarrow \mathbb{R}$.

(b) Evaluate $\int_0^1 s(x) dx$.

8. Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable function.

(a) Prove that if f' is Riemann integrable over $[a, b]$, then

$$\int_a^b f'(x) dx = f(b) - f(a).$$

(b) Give an example of f such that f' is not Riemann integrable.

9. Let $A := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{x} = 1, \mathbf{y} \cdot \mathbf{x} = 0\}$, where “ \cdot ” is the standard dot product in \mathbb{R}^3 (note that A can be naturally identified with the set of all tangent vectors to the unit sphere in \mathbb{R}^3). Show that, as a subset of \mathbb{R}^6 , the set A is locally the graph of a C^∞ map $\mathbb{R}^4 \rightarrow \mathbb{R}^2$ everywhere, i.e., at every point $p = (a_1, a_2, a_3, a_4, a_5, a_6) \in A$, there exist $1 \leq j_1 < j_2 \leq 6$ and C^∞ functions f, g defined in a neighborhood of $(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}) \in \mathbb{R}^4$, where $\{i_1, i_2, i_3, i_4\} = \{1, \dots, 6\} \setminus \{j_1, j_2\}$, with

$$f(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}) = a_{j_1},$$

$$g(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}) = a_{j_2},$$

and such that in a neighborhood of p , the set A is the graph

$$(x_{j_1}, x_{j_2}) = (f(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}), g(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})).$$

10. Let \mathbf{F} be the vector field in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ defined by

$$\mathbf{F}(x, y, z) := \frac{xz\mathbf{j} - xy\mathbf{k}}{(y^2 + z^2)\sqrt{x^2 + y^2 + z^2}}.$$

(a) Show that the curl of \mathbf{F} is given by

$$\nabla \times \mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

(b) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the unit circle centered at the point $(1, 1, 1)$ that lies on the plane $x + y + z = 3$ and has the orientation from the point $\left(1 - \frac{1}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}, 1 + \frac{2}{\sqrt{6}}\right)$ to $\left(1 - \frac{1}{\sqrt{6}}, 1 + \frac{2}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}\right)$ to $\left(1 + \frac{2}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}\right)$ and back to $\left(1 - \frac{1}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}, 1 + \frac{2}{\sqrt{6}}\right)$.

TIER I ANALYSIS EXAM
AUGUST 2013

Solve each of the following nine problems on a separate and clearly labeled sheet of paper. Fully justify your answers.

Notation:

- \mathbb{R} is the set of real numbers
 - \mathbb{R}^n is Euclidean space
 - $|x|$ is the Euclidean length of a vector $x \in \mathbb{R}^n$; absolute value when $n = 1$.
- (1) Fix positive integers n, N and a bounded set $A \subset \mathbb{R}^n$. We use the notation

$$\overline{B}(a, r) = \{x \in \mathbb{R}^n : |x - a| \leq r\}, \quad a \in \mathbb{R}^n, r \geq 0.$$

Show that there exist $a_1, a_2, \dots, a_N \in \mathbb{R}^n$ and numbers $r_1, \dots, r_N \in [0, +\infty)$ such that

$$A \subset \bigcup_{k=1}^N \overline{B}(a_k, r_k)$$

and the sum $\sum_{k=1}^N r_k^2$ is as small as possible. In other words, the set $\{\sum_{k=1}^N r_k^2 : A \text{ can be covered with a collection } (\overline{B}(a_k, r_k))_{k=1}^N\}$ has a smallest element.

- (2) Is the sequence $(\cos(\pi\sqrt{n^2 + n}))_{n=1}^{\infty}$ convergent?
(3) For which values of $x \in \mathbb{R}$ does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{x + n}$$

converge? Is the convergence uniform on the interval $(-1, 1)$?

- (4) Consider the functions $f_n : [0, 1] \rightarrow \mathbb{R}$ defined by $f_n(x) = (1 - x^n)^{2^n}$ for $x \in [0, 1]$ and $n \in \mathbb{N}$. Prove that the limit $\lim_{n \rightarrow \infty} f_n(x)$ exists for every $x \in [0, 1]$. Is the convergence uniform on $[0, 1]$?
(5) Let $f : [0, 1] \rightarrow \mathbb{R}$ be a Riemann integrable function and let $\varepsilon > 0$. Show that there exist continuous functions $g, h : [0, 1] \rightarrow \mathbb{R}$ such that $g(x) \leq f(x) \leq h(x)$ for all $x \in [0, 1]$, and

$$\int_0^1 (h(x) - g(x)) dx < \varepsilon.$$

Is the converse statement true?

- (6) Assume the function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the property

$$f(x + t) \geq f(x) - t^2$$

for all real values of x and all positive values of t . Prove that f must be nondecreasing.

- (7) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be everywhere differentiable, and assume that the Jacobian of f is not singular at any point $x = (x_1, x_2) \in \mathbb{R}^2$. Assume that $|f(x)| \leq 1$ whenever $|x| = 1$, and prove that in fact $|f(x)| \leq 1$ whenever $|x| \leq 1$.

(8) Compute

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-|x-y|^2}}{1+|x+y|^2} dx dy.$$

(9) Given a positive number $r \neq 1$, set $C_r = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = r\}$. Calculate the line integral

$$\int_{C_r} \frac{x dy - y dx}{x^2 + y^2},$$

where C_r is oriented counterclockwise relative to $(1, 0)$.

Tier 1 Analysis Exam

JANUARY 6, 2014

Each problem below is worth 10 points. Answer each one on a new sheet of paper, writing the problem number on every sheet. Use only one side of each sheet, and fully justify all answers. Put your answers in the correct order when you turn them in. You have 4 hours.

0.1. Suppose a metric space (X, d) has this property: Given any $\varepsilon > 0$, there is a non-empty finite subset $X_\varepsilon \subset X$ such that for every $x \in X$, we have

$$\inf\{d(x, p) : p \in X_\varepsilon\} \leq \varepsilon$$

- a) Show that in this case, every sequence in X has a Cauchy subsequence.
- b) Give an example showing that (a) fails if we don't require the X_ε 's to be finite.

0.2. For $p, q \in \mathbf{R}^3$, let $|p|$ and $p \times q$ respectively denote the euclidean norm of p , and the cross-product of p and q . Define $d : \mathbf{R}^3 \times \mathbf{R}^3 \rightarrow [0, \infty)$ by

$$d(p, q) = \begin{cases} |p| + |q|, & p \times q \neq 0 \\ |p - q|, & p \times q = 0 \end{cases}$$

- a) Show that d is a metric on \mathbf{R}^3 .
- b) Show that the closed unit d -ball centered at $(0, 0, 0)$ is **not** d -compact.
- c) Show that the closed unit d -ball centered at $(1, 1, 1)$ is d -compact.

0.3. Assume $f, \omega : \mathbf{R} \rightarrow \mathbf{R}$ are functions, with $\omega(0) = 0$. Assume too that for some $\alpha > 1$, we have

$$(1) \quad f(b) \leq f(a) + \omega(|b - a|)^\alpha \quad \text{for all } a, b \in \mathbf{R}$$

- a) Show that when ω is differentiable at $x = 0$, our assumptions make f infinitely differentiable at every point.
- b) Give an example showing that when $\alpha > 1$ but ω is merely continuous, our assumptions do not force differentiability of f at all points.

0.4. Show that every sequence in \mathbf{R} has a weakly monotonic (i.e. non-increasing, or non-decreasing) subsequence.

0.5. Show that the series converges, but not absolutely:

$$\sum_{n=1}^{\infty} \left(\exp \left(\frac{(-1)^n}{n} \right) - 1 \right)$$

0.6. Consider this integral:

$$\int_0^{\infty} \sin(x^p) dx$$

- a) Does it converge when $p = 1$?
- b) Does it converge when $p < 0$?
- c) Does it converge when $p > 1$?

0.7. Suppose $f : [0, \infty) \rightarrow [0, \infty)$ is a continuous bijection and consider the series

$$\sum_{n=1}^{\infty} \frac{nf(x^2)}{1 + n^3 f(x^2)^2}$$

- a) Show that the series converges pointwise for all $x \in \mathbf{R}$.
- b) Show that it converges *uniformly* on $[\varepsilon, \infty)$ when $\varepsilon > 0$.
- c) Show that it does *not* converge uniformly on \mathbf{R} .

0.8. Let S denote the upper hemisphere of radius $r > 0$ centered at $\mathbf{0} \in \mathbf{R}^3$, i.e.,

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = r^2 \text{ and } z \geq 0\}$$

and suppose $F : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is the vector field given by

$$F(x, y, z) = \begin{pmatrix} xy^2 \tanh(x^2 + z) \\ x + y^4 \sin(z) e^{-x^2} \\ x^2(x^3 + 3)y e^{-x^2 - y^2 - z^2} \end{pmatrix}.$$

Compute

$$\int_S \operatorname{curl}(F) \cdot \mathbf{n} \, dS$$

where \mathbf{n} is the upward pointing unit surface normal, and dS is the area element on S .

0.9. Consider this system of equations in the variables u, v, s, t :

$$(uv)^4 + (u + s)^3 + t = 0$$

$$\sin(uv) + e^v + t^2 - 1 = 0.$$

Prove that near the origin $\mathbf{0} \in \mathbf{R}^4$, its solutions form the graph of a continuously differentiable function $G : \mathbf{R}^2 \rightarrow \mathbf{R}^2$. Clearly indicate the dependent and independent variables.

0.10. Let

$$f(x, y) = \begin{cases} \frac{yx^6 + y^3 + x^3y}{x^6 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

- a) Show that all directional derivatives of f exist at $(0, 0)$, and depend linearly on the vector we differentiate along.
- b) Show that nevertheless, f is *not* differentiable at $(0, 0)$.

Tier I Analysis Exam, August 2014

Try to work all questions. Providing justification for your answers is crucial.

1. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable with $f(0) = f(1) = 0$ and

$$\{x : f'(x) = 0\} \subset \{x : f(x) = 0\} .$$

Show that $f(x) = 0$ for all $x \in [0, 1]$.

2. Let (a_n) be a bounded sequence for $n = 1, 2, \dots$ such that

$$a_n \geq (1/2)(a_{n-1} + a_{n+1}) \text{ for } n \geq 2 .$$

Show that (a_n) converges.

3. Suppose $K \subset \mathbb{R}^n$ is a compact set and $f : K \rightarrow \mathbb{R}$ is continuous. Let $\varepsilon > 0$ be given. Prove that there exists a positive number M such that for all x and y in K one has the inequality:

$$|f(x) - f(y)| \leq M \|x - y\| + \varepsilon .$$

Here $\|\cdot\|$ denotes the Euclidean norm in \mathbb{R}^n . Then give a counter-example to show that the inequality is not in general true if one takes $\varepsilon = 0$.

4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth function and let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$g(x_1, \dots, x_n) = x_1^5 + \dots + x_n^5 .$$

Suppose $g \circ f \equiv 0$. Show that $\det Df \equiv 0$.

5. The point $(1, -1, 2)$ lies on both the surface described by the equation

$$x^2(y^2 + z^2) = 5$$

and on the surface described by

$$(x - z)^2 + y^2 = 2 .$$

Show that in a neighborhood of this point, the intersection of these two surfaces can be described as a smooth curve in the form $z = f(x)$, $y = g(x)$. What is the direction of the tangent to this curve at $(1, -1, 2)$?

6. For what smooth functions $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is there a smooth vector field $\mathbf{W}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\text{curl } \mathbf{W} = \mathbf{V}$, where

$$\mathbf{V}(x, y, z) = (y, x, f(x, y, z))?$$

For f in this class, find such a \mathbf{W} . Is it unique?

7. For each positive integer n let $f_n: [0, 1] \rightarrow \mathbb{R}$ be a continuous function, differentiable on $(0, 1]$, such that

$$|f'_n(x)| \leq \frac{1 + |\ln x|}{\sqrt{x}} \quad \text{for } 0 < x \leq 1.$$

and such that

$$-10 \leq \int_0^1 f_n(x) dx \leq 10.$$

Prove that $\{f_n\}$ has a uniformly convergent subsequence on $[0, 1]$.

8. Define for $n \geq 2$ and $p > 0$

$$H_n(p) = \sum_{k=1}^n (\log k)^p \quad \text{and} \quad a_n(p) = \frac{1}{H_n(p)}.$$

For which p does $\sum_n a_n(p)$ converge?

9. Given any continuous, piecewise smooth curve $\gamma: [0, 1] \rightarrow \mathbb{R}^2$, consider the following notion of its 'length' \tilde{L} defined through the line integral:

$$\tilde{L}(\gamma) := \int_{\gamma} |x| ds = \int_0^1 |x(t)| \sqrt{x'(t)^2 + y'(t)^2} dt$$

where a point in \mathbb{R}^2 is written as (x, y) and $\gamma(t) = (x(t), y(t))$.

- (a) Suppose we define a notion of distance \tilde{d} between two points p_1 and p_2 in \mathbb{R}^2 via

$$\tilde{d}(p_1, p_2) := \inf\{\tilde{L}(\gamma) : \gamma(0) = p_1, \gamma(1) = p_2\}.$$

Working through the definition of metric, determine which properties of a metric hold for \tilde{d} , and which, if any, do not.

- (b) Determine the value of $\tilde{d}((1, 1), (-1, -2))$ and determine a curve achieving this infimum.

Tier 1 Analysis Exam

JANUARY 5, 2015

You have 4 hours to work these 10 problems. Each is worth 10 points.

- Start each answer on a clean sheet of paper
- Use only one side of each sheet
- Circle the prob. number in the upper-right corner of each sheet
- Fully justify all answers.
- Put your answers in the correct order before submitting them.

0.1. An open set $U \subset \mathbf{R}^n$ contains the closed origin-centered unit ball $B = B(\mathbf{0}, 1)$. If a C^1 mapping $f: U \rightarrow \mathbf{R}^n$ with rank n obeys $\|f(x) - x\| < 1/2$ for all $x \in U$, show that

- $\|f\|^2$ must attain a minimum in the interior of B .
- $f(p) = \mathbf{0}$ for some $p \in B$.

0.2. Suppose $f, g: \mathbf{R} \rightarrow \mathbf{R}$, are functions that obey

$$f(x+h) = f(x) + g(x)h + a(x, h)$$

for all $x, h \in \mathbf{R}$, with $|a(x, h)| \leq Ch^3$ for some constant C .

Show that f is affine (i.e., $f(x) = mx + b$ for some $m, b \in \mathbf{R}$).

0.3. Suppose f is differentiable on an open interval containing $[-1, 1]$. Do **not** assume continuity of f' .

- Supposing $f'(-1)f'(1) < 0$ show that $f'(x) = 0$ for some $x \in (-1, 1)$.
- Supposing that $f'(-1) < L < f'(1)$ for some $L \in \mathbf{R}$, show that $f'(x) = L$ for some $x \in (-1, 1)$.

0.4. Suppose (X, d) is a complete metric space. Show that if every continuous function on a subset $U \subset X$ attains a minimum, then U is closed.

0.5. Define the distance from a point p in a metric space (X, d) to a subset $Y \subset X$ by

$$d(p, Y) := \inf\{d(x, y) : y \in Y\}$$

For any $\varepsilon > 0$, define

$$Y_\varepsilon = \{x \in X : d(x, Y) \leq \varepsilon\}$$

Finally, given any two **bounded** sets $A, B \subset X$, define

$$d_S(A, B) = \inf\{\varepsilon > 0: A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\}$$

(a) Show that d_S yields a metric on the set of **closed bounded** subsets of X .

(b) Show that d_S fails to do so on the set of **bounded** subsets of X .

0.6. Determine whether the series converges or not.

$$\sum_{j=1}^{\infty} \left(e^{(-1)^j \sin(1/j)} - 1 \right)$$

0.7. Let B_r denote the ball $|x| \leq r$ in \mathbf{R}^3 , and write dS_r for the area element on its boundary ∂B_r .

The electric field associated with a uniform charge distribution on ∂B_R may be expressed as

$$E(x) = C \int_{\partial B_R} \nabla_x |x - y|^{-1} dS_y,$$

a) Show that for any $r < R$, the electric flux $\int_{\partial B_r} E(x) \cdot \nu dS_x$ through ∂B_r equals zero.

b) Show that $E(x) \equiv 0$ for $|x| < R$ (“a conducting spherical shell shields its interior from outside electrical effects”).

0.8. Let Q be a bounded closed rectangle in \mathbf{R}^n , and suppose we have functions $f, g: Q \rightarrow \mathbf{R}$ that, for some $K > 0$, satisfy

$$|f(x) - f(y)| \leq K |g(x) - g(y)|$$

and all $x, y \in Q$. Prove that if g is Riemann integrable, then so is f . Deduce further that integrability of f implies that of $|f|$.

0.9. Suppose $f: U \rightarrow \mathbf{R}$ is a differentiable function defined on an open set $U \supset [0, 1]^2$. Assuming $f(0, 0) = 3$ and $f(1, 1) = 1$, prove that for $|\nabla f| \geq \sqrt{2}$ somewhere in U .

0.10. Consider this quadratic system in \mathbf{R}^4 :

$$\begin{aligned} a^2 + b^2 - c^2 - d^2 &= 0 \\ ac + bd &= 0 \end{aligned}$$

Show the system can be solved for (a, c) in terms of (b, d) (or vice-versa) near any solution $(a_0, b_0, c_0, d_0) \neq (0, 0, 0, 0)$. (You need not find explicit solutions here.)

**Analysis Tier I Exam
August 2015**

- **Be sure to fully justify all answers.**
 - **Scoring:** Each problem is worth 10 points.
 - **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
 - Please be sure that you assemble your test with the problems presented in correct order.
-

1. Let $f(x)$ be a continuous function on $(0, 1]$ and

$$\liminf_{x \rightarrow 0^+} f(x) = \alpha, \quad \limsup_{x \rightarrow 0^+} f(x) = \beta.$$

Prove that for any $\xi \in [\alpha, \beta]$, there exist $\{x_n \in (0, 1] \mid n = 1, 2, \dots\}$ such that

$$\lim_{n \rightarrow \infty} f(x_n) = \xi.$$

2. Let $f(x)$ be a function which is defined and is continuously differentiable on an open interval containing the closed interval $[a, b]$, and let

$$f^{-1}(0) = \{x \in [a, b] \mid f(x) = 0\}.$$

Assume that $f^{-1}(0) \neq \emptyset$, and for any $x \in f^{-1}(0)$, $f'(x) \neq 0$. **Prove** the following assertions:

- (a) $f^{-1}(0)$ is a finite set;
- (b) Let p be the number of points in $f^{-1}(0)$ such that $f'(x) > 0$, and q be the number of points in $f^{-1}(0)$ such that $f'(x) < 0$. Then

$$|p - q| \leq 1.$$

3. Let $\sum_{n=1}^{\infty} a_n$ be a convergent positive term series ($a_n \geq 0$ for all n). **Show** that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. **Is the converse true?**

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable with f' uniformly continuous. Suppose $\lim_{x \rightarrow \infty} f(x) = L$ for some L . Does $\lim_{x \rightarrow \infty} f'(x)$ exist?
5. Let $E \subset \mathbb{R}$ be a set with the property that any countable family of closed sets that cover E contains a finite subcollection which covers E . **Show** that E must consist of finitely many points.
6. Suppose that a function $f(x)$ is defined as the sum of a series:

$$f(x) = 1 - \frac{1}{(2!)^2}(2015x)^2 + \frac{1}{(4!)^2}(2015x)^4 - \frac{1}{(6!)^2}(2015x)^6 + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{((2k)!)^2} (2015x)^{2k}.$$

Evaluate

$$\int_0^{\infty} e^{-x} f(x) dx.$$

7. **Find** the volume of the solid S in \mathbb{R}^3 , which is the intersection of two cylinders $C_1 = \{(x, y, z) \in \mathbb{R}^3; y^2 + z^2 \leq 1\}$ and $C_2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + z^2 \leq 1\}$.
8. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be continuous. Suppose that f has the property that for any compact set $K \subset \mathbb{R}^m$, the set $f^{-1}(K) \subset \mathbb{R}^n$ is bounded. **Prove** that $f(\mathbb{R}^n)$ is a closed subset of \mathbb{R}^m , **or give** a counterexample to this claim.
9. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ have continuous second-order partial derivatives. **Find all points** where the condition in the implicit function theorem is satisfied so that $F(x - y, y - z) = 0$ defines an implicit function $z = z(x, y)$, and **derive** explicit formulas, in terms of partial derivatives of F , for

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x \partial y}.$$

10. Suppose that a monotone sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a continuous function F on some closed interval $[a, b]$. **Prove** that the convergence is uniform.

Note: In this problem by a monotone sequence of functions we mean a sequence f_n such that either $f_n(x) \leq f_{n+1}(x)$ for all n and all $x \in [a, b]$, or $f_n(x) \geq f_{n+1}(x)$ for all n and all $x \in [a, b]$.

TIER I ANALYSIS EXAM, JANUARY 2016

Solve all nine problems. They all count equally. Show all computations.

1. Let $a > 0$ and let x_n be a sequence of real numbers. Assume the sequence

$$y_n = \frac{x_1 + x_2 + \dots + x_n}{n^a}$$

is bounded. Show that for each $b > a$, the series

$$\sum_{n=1}^{\infty} \frac{x_n}{n^b}$$

is convergent.

2. (a) Show that for each integer $n \geq 1$ there exists exactly one $x > 0$ such that

$$\frac{1}{\sqrt{nx+1}} + \frac{1}{\sqrt{nx+2}} + \dots + \frac{1}{\sqrt{nx+n}} = \sqrt{n}.$$

- (b) Call x_n the solution from (a). Find

$$\lim_{n \rightarrow \infty} x_n.$$

3. Let (X, d) be a compact metric space and let ρ be another metric on X such that

$$\rho(x, x') \leq d(x, x'), \text{ for all } x, x' \in X.$$

Show that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\rho(x, x') < \delta \implies d(x, x') < \epsilon.$$

4. Prove that for each $x \in \mathbb{R}$ there is a choice of signs $s_n \in \{-1, 1\}$ such that the series

$$\sum_{n=1}^{\infty} \frac{s_n}{\sqrt{n}}$$

converges to x .

5. Assume the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the property

$$f(x+t, y+s) \geq f(x, y) - s^2 - t^2,$$

for each $(x, y) \in \mathbb{R}^2$ and each $(s, t) \in \mathbb{R}^2$. Prove that f must be constant.

6. Assume $f : [0, 1] \rightarrow \mathbb{R}$ is continuous and $f(0) = 2016$. Find

$$\lim_{n \rightarrow \infty} \int_0^1 f(x^n) dx.$$

7. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be two differentiable functions with $f(x, y, z) = g(xy, yz)$ and suppose that $g(u, v)$ satisfies

$$g(2, 6) = 2, \quad \frac{\partial g}{\partial u}(2, 6) = -1, \quad \text{and} \quad \frac{\partial g}{\partial v}(2, 6) = 3.$$

Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 2\}$ admits a tangent plane at the point $(1, 2, 3)$, and find an equation for it.

8. Let \mathcal{C} be the collection of all positively oriented (i.e. counter-clockwise) simple closed curves C in the plane. Find

$$\sup\left\{\int_C (y^3 - y)dx - 2x^3dy : C \in \mathcal{C}\right\}.$$

Is the supremum attained?

9. Let

$$H = \{(x, y, z) \mid z > 0 \text{ and } x^2 + y^2 + z^2 = R^2\}$$

be the upper hemisphere of the sphere of radius R centered at the origin in \mathbb{R}^3 . Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the vector field

$$F(x, y, z) = \left(x^2 \sin(y^2 - z^3), xy^4z + y, e^{-x^2 - y^2} + yz\right)$$

Find $\int_H F \cdot \hat{n} \, dS$ where \hat{n} is the outward pointing unit surface normal and dS is the area element.

TIER 1 ANALYSIS EXAM, AUGUST 2016

Directions: Be sure to use **separate** pieces of paper for different solutions. This exam consists of nine questions and each counts equally. Credit may be given for partial solutions.

- (1) Let $f : [0, 1] \rightarrow \mathbb{R}$ be a nondecreasing function, and let D be the set of $x \in [0, 1]$ such that f is not continuous at x . Is the set D necessarily compact? Fully justify your answer.
- (2) Show that there exist a real number $\varepsilon > 0$ and a differentiable function $f : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ such that

$$e^{x^2+f(x)} = 1 - \sin(x + f(x)).$$

- (3) Prove that the function f defined by

$$f(x) := \sum_{n=0}^{\infty} \frac{\cos(n^2x)}{2^{nx}}$$

is continuous on the interval $(0, \infty)$.

- (4) Using only the definitions of continuity and (sequential) compactness, prove that if $K \subset \mathbb{R}$ is (sequentially) compact and $f : K \rightarrow \mathbb{R}$ is continuous, then f is uniformly continuous, that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.
- (5) Show that if $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = 0$, then the set of limit points of $\{x_n\}$ is connected, that is, either empty, a single point, or an interval.

- (6) Let a and b be positive numbers, and let Γ be the closed curve in \mathbb{R}^3 that is the intersection of the surface $\{(x, y, z) : z = b \cdot x \cdot y\}$ and the cylinder $\{(x, y, z) : x^2 + y^2 = a^2\}$. Let r be a parametrization of Γ so that the curve is oriented counter-clockwise when looking down upon it from high up on the z -axis. Compute

$$\int_{\Gamma} F \cdot dr.$$

where F is the vector valued function defined by $F(x, y, z) = (y, z, x)$.

(7) Let $\Omega = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, and define $f : \Omega \rightarrow \mathbb{R}$ by

$$f(x, y) = \frac{2 + \sqrt{(1+x)^2 + y^2} + \sqrt{(1-x)^2 + y^2}}{\sqrt{y}}.$$

Show that f has achieves its minimum value on Ω at a unique point $(x_0, y_0) \in \Omega$ and find (x_0, y_0) .

(8) Suppose that $(a_n)_{n=1}^{\infty}$ is a bounded sequence of positive numbers. Show that

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0$$

if and only if

$$\lim_{n \rightarrow \infty} \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} = 0.$$

(9) Define $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$d(x, y) = \frac{\|x - y\|}{\|x\|^2 + \|y\|^2 + 1}$$

where $\|x\|^2 = x_1^2 + \cdots + x_n^2$. Let $A \subset \mathbb{R}^n$ be such that there exists $\epsilon > 0$ so that if $a, b \in A$ with $a \neq b$, then $d(a, b) \geq \epsilon$. Show that A is finite.

Tier 1 Analysis Exam

January 2017

Do all nine problems. They all count equally. Show your work and justify your answers.

1. Define a subset X of \mathbb{R}^n to have property \mathcal{C} if every sequence with exactly one accumulation point in X converges in X . (Recall that x is an accumulation point of a sequence (x_n) if every neighborhood of x contains infinitely many x_n .)
 - (a) Give an example of a subset $X \subset \mathbb{R}^n$, for some $n \geq 1$, that does not have property \mathcal{C} , together with an example of a non-converging sequence in X with exactly one accumulation point.
 - (b) Show that any subset X of \mathbb{R}^n satisfying property \mathcal{C} is compact.

2. Prove that the sequence

$$a_1 = 1, \quad a_2 = \sqrt{7}, \quad a_3 = \sqrt{7\sqrt{7}}, \quad a_4 = \sqrt{7\sqrt{7\sqrt{7}}}, \quad a_5 = \sqrt{7\sqrt{7\sqrt{7\sqrt{7}}}}, \quad \dots$$

converges, then find its limit.

3. Given any metric space (X, d) show that $\frac{d}{1+d}$ is also a metric on X , and show that $(X, \frac{d}{1+d})$ shares the same family of metric balls as (X, d) .
4. Suppose that a function $f(x)$ is defined as the sum of series

$$f(x) = \sum_{n \geq 3} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \sin(nx).$$

- (a) Explain why $f(x)$ is continuous.
- (b) Evaluate

$$\int_0^\pi f(x) dx.$$

5. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function with $h(0) = 0$, and consider the following system of equations:

$$\begin{aligned} e^x + h(y) &= u^2, \\ e^y - h(x) &= v^2. \end{aligned}$$

Show that there exists a neighborhood $V \subset \mathbb{R}^2$ of $(1, 1)$ such that for each $(u, v) \in V$ there is a solution $(x, y) \in \mathbb{R}^2$ to this system.

6. Let n be a positive integer. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function. Assume that $f(\vec{x}) \rightarrow 0$ whenever $\|\vec{x}\| \rightarrow \infty$. Show that f is uniformly continuous on \mathbb{R}^n .

7. Let $f_n(x)$ and $f(x)$ be continuous functions on $[0, 1]$ such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in [0, 1]$. Answer each of the following questions. If your answer is “yes”, then provide an explanation. If your answer is “no”, then give a counterexample.

(a) Can we conclude that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx?$$

(b) If in addition we assume $|f_n(x)| \leq 2017$ for all n and for all $x \in [0, 1]$, can we conclude that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx?$$

8. Evaluate the flux integral $\iint_{\partial V} \vec{F} \cdot \vec{n} \, dS$, where the field \vec{F} is

$$\vec{F}(x, y, z) = (xe^{xy} - 2xz + 2xy \cos^2 z) \vec{i} + (y^2 \sin^2 z - ye^{xy} + y) \vec{j} + (x^2 + y^2 + z^2) \vec{k},$$

and V is the (bounded) solid in \mathbb{R}^3 bounded by the xy -plane and the surface $z = 9 - x^2 - y^2$, ∂V is the boundary surface of V , and \vec{n} is the outward pointing unit normal vector on ∂V .

9. A continuously differentiable function f from $[0, 1]$ to $[0, 1]$ has the properties

(a) $f(0) = f(1) = 0$;

(b) $f'(x)$ is a non-increasing function of x .

Prove that the arclength of the graph of f does not exceed 3.

Tier I Analysis Exam

August, 2017

- **Be sure to fully justify all answers.**
 - **Scoring:** Each problem is worth 10 points.
 - **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
 - Please be sure that you assemble your test with the problems presented in correct order.
-

- (1) Let X be the set of all functions $f : \mathbb{N} \rightarrow \{0, 1\}$, taking only two values 0 and 1. Define the metric d on X by

$$d(f, g) = \begin{cases} 0 & \text{if } f = g, \\ \frac{1}{2^m} & \text{if } m = \min\{n \mid f(n) \neq g(n)\}. \end{cases}$$

- (a) **Prove** that (X, d) is compact.
(b) **Prove** that no point in (X, d) is isolated.
- (2) Let $C[0, 1]$ be the space of all real continuous functions defined on the interval $[0, 1]$. Define the distance on $C[0, 1]$ by

$$d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|.$$

Prove that the following set $\mathcal{S} \subset C[0, 1]$ is not compact:

$$\mathcal{S} = \{f \in C[0, 1] \mid d(f, 0) = 1\},$$

where $0 \in C[0, 1]$ stands for the constant function with value 0.

- (3) Let $F(x, y) = \sum_{n=1}^{\infty} \sin(ny) \cdot e^{-n(x+y)}$. **Prove that** there are a $\delta > 0$ and a unique differentiable function $y = \varphi(x)$ defined on $(1 - \delta, 1 + \delta)$, such that

$$\varphi(1) = 0, \quad F(x, \varphi(x)) = 0 \quad \forall x \in (1 - \delta, 1 + \delta).$$

- (4) **Prove or find** a counterexample: if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable with $f(0) = 0$, then there exist continuous functions $g_1, \dots, g_n : \mathbb{R}^n \rightarrow \mathbb{R}$ with

$$f(x) = x_1 g_1(x_1, \dots, x_n) + \cdots + x_n g_n(x_1, \dots, x_n).$$

- (5) Let $\{f_n\}$ be a sequence of real-valued, concave functions defined on an open interval $(-a, a)$ ($-f_n$ is convex). Let $g : (-a, a) \rightarrow \mathbb{R}$. Suppose f_n and g are differentiable at 0,

$$\liminf f_n(t) \geq g(t) \text{ for all } t, \text{ and } \lim f_n(0) = g(0).$$

Show that $\lim f'_n(0) = g'(0)$.

- (6) Let $f(x, y) = \frac{x^2 y}{x^4 + y^2}$ for $(x, y) \neq (0, 0)$.
- (a) Can f be defined at $(0, 0)$ so that $f_x(0, 0)$ and $f_y(0, 0)$ exist? **Justify** your answer.
- (b) Can f be defined at $(0, 0)$ so that f is differentiable at $(0, 0)$? **Justify** your answer.

- (7) Let $f : [-1, 1] \rightarrow \mathbb{R}$ with f, f', f'', f''' being continuous. **Show** that

$$\sum_{n=2}^{\infty} \left\{ n \left[f\left(\frac{1}{n}\right) - f\left(-\frac{1}{n}\right) \right] - 2f'(0) \right\}$$

converges absolutely.

- (8) Let $\{f_n\}$ be a uniformly bounded sequence of continuous real-valued functions on a closed interval $[a, b]$, and let $g_n(x) = \int_a^x f_n(t) dt$ for each $x \in [a, b]$. **Show** that the sequence of functions $\{g_n\}$ contains a uniformly convergent subsequence on $[a, b]$.

- (9) **Compute** $\int_D x dx dy$, where $D \subset \mathbb{R}^2$ is the region bounded by the curves $x = -y^2$, $x = 2y - y^2$, and $x = 2 - 2y - y^2$. **Show** your work.

- (10) Let

$$x_0 > 0, \quad x_{n+1} = \frac{1}{2} \left(x_n + \frac{4}{x_n} \right), \quad n = 0, 1, 2, 3, \dots$$

Show that $x = \lim_{n \rightarrow \infty} x_n$ exists, and **find** x .

Tier I Analysis January 2018

Problem 1. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \rightarrow Y$ be surjective such that

$$\frac{1}{2}d_X(x, y) \leq d_Y(f(x), f(y)) \leq 2d_X(x, y)$$

for all $x, y \in X$. Show that if (X, d_X) is complete, then also (Y, d_Y) is complete.

Problem 2. Show that

$$\lim_{n \rightarrow \infty} \left(2\sqrt{n} - \sum_{k=1}^n \frac{1}{\sqrt{k}} \right)$$

exists.

Problem 3. Assume that *bitter* is a property of subsets of $[0, 1]$ such that the union of two bitter sets is bitter. Subsets of $[0, 1]$ that are not bitter are called *sweet*. Thus every subset of $[0, 1]$ is either bitter or sweet. A *sweet spot* of a set $A \subset [0, 1]$ is a point $x_0 \in [0, 1]$ such that for every open set $U \subset \mathbb{R}$ that contains x_0 , the set $A \cap U$ is sweet. Show that if $A \subset [0, 1]$ is sweet, then A has a sweet spot.

Problem 4. Let f and g be periodic functions defined on \mathbb{R} , not necessarily with the same period. Suppose that

$$\lim_{x \rightarrow \infty} f(x) - g(x) = 0 .$$

Show that $f(x) = g(x)$ for all x .

Problem 5. Let $0 < x_n < 1$ be an infinite sequence of real numbers such that for all $0 < r < 1$

$$\sum_{x_n < r} \log \frac{r}{x_n} \leq 1 .$$

Show that

$$\sum_{n=1}^{\infty} (1 - x_n) < \infty .$$

Problem 6. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally. Show that the series

$$\sum_{n=3}^{\infty} n(\log n)(\log \log n)^2 a_n$$

diverges.

Problem 7. Find the absolute minimum of the function $f(x, y, z) = xy + yz + zx$ on the set $g(x, y, z) = x^2 + y^2 + z^2 = 12$.

Problem 8. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^1 map such that $f^{-1}(y)$ is a finite set for all $y \in \mathbb{R}^2$. Show that the determinant $\det df(x)$ of the Jacobi matrix of f cannot vanish on an open subset of \mathbb{R}^2 .

Problem 9. A *regular surface* is given by a continuously differentiable map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ so that the differential $df_x : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ has rank 2 for all $x \in \mathbb{R}^2$. The *tangent plane* T_x is the 2-dimensional subspace $df_x(\mathbb{R}^2) \subset \mathbb{R}^3$. Assume that a vector field X in \mathbb{R}^3 is orthogonal to T_x for all x , i.e. $X(f(x)) \cdot Y = 0$ for all $x \in \mathbb{R}^2$ and all $Y \in T_x$. Show that $X \cdot (\nabla \times X) = 0$ at all points $f(x)$.

Problem 10. Let $f(x, y)$ be a function defined on \mathbb{R}^2 such that

- For any fixed x , the function $y \mapsto f(x, y)$ is a polynomial in y ;
- For any fixed y , the function $x \mapsto f(x, y)$ is a polynomial in x .

Show that f is a polynomial, i.e.

$$f(x, y) = \sum_{i,j=0}^N a_{ij} x^i y^j$$

with suitable $a_{i,j} \in \mathbb{R}$, $i, j = 0, \dots, N$.

TIER I ANALYSIS EXAMINATION

August 2018

Instructions: There are ten problems, each of equal value. Show your work, justifying all steps by direct calculation or by reference to an appropriate theorem.

Notation: For $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, $|\mathbf{x}| = \sqrt{x_1^2 + \dots + x_n^2}$, and $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$.

1. Suppose $(a_n)_{n=1}^\infty$ is a sequence of positive real numbers and $\sum_{n=1}^\infty a_n = \infty$. Prove that there exists a sequence of positive real numbers $(b_n)_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} b_n = 0$ and $\sum_{n=1}^\infty a_n b_n = \infty$.

2. Show that $\sum_{n=1}^\infty \sin(x^n)/n!$ converges uniformly for $x \in \mathbb{R}$ to a C^1 function $f : \mathbb{R} \rightarrow \mathbb{R}$, and compute an expression for the derivative. Justify this computation.

3. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be differentiable. Show that the intersection of all tangent planes to the surface $z = xf(x/y)$ ($x, y \in (0, \infty)$) is nonempty.

4. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer that is less than or equal to x . Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{n} \rfloor}}{n}$$

converges. Suggestion: The inequality

$$\frac{1}{\ell+1} < \int_{\ell}^{\ell+1} \frac{1}{x} dx < \frac{1}{\ell}$$

might be helpful. You do not need to justify this inequality.

5. Let B be the closed unit ball in \mathbb{R}^2 with respect to the usual metric, d (defined above). Let ρ be the metric on B defined by

$$\rho(x, y) = \begin{cases} |\mathbf{x} - \mathbf{y}| & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are on the same line through the origin,} \\ |\mathbf{x}| + |\mathbf{y}| & \text{otherwise,} \end{cases}$$

for $\mathbf{x}, \mathbf{y} \in B$. (Note that $\rho(x, y)$ is the minimum distance travelled in the usual metric in going from x to y along lines through the origin.) Suppose $f : B \rightarrow \mathbb{R}$ is a function that is uniformly continuous on B with respect to the metric ρ on B and the usual metric on \mathbb{R} . Prove that f is bounded.

6. Let

$$f(x) := \begin{cases} \sin x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove or disprove: there exists $\epsilon > 0$ such that f is invertible when restricted to $(-\epsilon, \epsilon)$.

7. Define a sequence of functions $f_n : [0, 2\pi] \subset \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_n(x) = e^{\sin(nx)},$$

and define $F_n(x) = \int_0^x f_n(y) dy$. Show that there exists a subsequence $(F_{n_k})_{k=1}^\infty$ of $(F_n)_{n=1}^\infty$ that converges uniformly on $x \in [0, 2\pi]$ to a continuous limit F_* .

8. Let a closed curve, γ , be parameterized by a function $f : [0, 1] \rightarrow \mathbb{R}^2$ with a continuous derivative and $f(0) = f(1)$. Suppose that

$$(1) \quad \int_\gamma (y^3 \sin^2 x dx - x^5 \cos^2 y dy) = 0.$$

Show that there exists a pair $\{x, y\} \neq \{0, 1\}$ with $x \neq y$ and $f(x) = f(y)$. Give an example of a curve satisfying (1) such that the only pairs $\{x, y\}$ with $x \neq y$ and $f(x) = f(y)$ are subsets of $\{0, 1/2, 1\}$.

9. Fix $a > 0$. Let S be the *half-ellipsoid* defined by $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z/a)^2 = 1 \text{ and } z \geq 0\}$. Let \mathbf{v} be the vector field given by $\mathbf{v}(x, y, z) = (x, y, z + 1)$, and let \mathbf{n} be the outward unit normal field to the ellipsoid $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z/a)^2 = 1\}$.

(a) From the fact that the volume of $D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \text{ and } z \geq 0\}$ is $2\pi/3$, which you may assume without proof, use the change-of-variables formula in \mathbb{R}^3 to find the volume of $E := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z/a)^2 \leq 1 \text{ and } z \geq 0\}$.

(b) Evaluate

$$\iint_S \mathbf{v} \cdot \mathbf{n} dA,$$

where dA denotes the surface area element.

10. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be C^2 , let I denote the $n \times n$ identity matrix, let

$$D^2 f(x) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n},$$

and assume that there exists a positive real number a such that $D^2 f(x) - aI$ is positive definite for all $x \in \mathbb{R}^n$, or equivalently, assume that there exists a positive real number a such that $D_{\mathbf{u}}[D_{\mathbf{u}} f](x) \geq a$ for all unit vectors $\mathbf{u} \in \mathbb{R}^n$ and points $x \in \mathbb{R}^n$, where $D_{\mathbf{u}}$ denotes the directional derivative in the direction \mathbf{u} . (You do not have to prove the equivalence of these two versions of the assumption.)

(a) Let ∇f denote the gradient of f . Show that there exists a point $x \in \mathbb{R}^n$ such that $\nabla f(x) = 0$.

(b) Show that the map $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is onto.

(c) Show that the map $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is globally invertible, and the inverse is C^1 .

TIER I ANALYSIS EXAM, JANUARY 2019

Solve all nine problems. They all count equally. Show all computations.

1. Let $f : \mathbb{R} \rightarrow [0, 1]$ be continuous. Let $x_1 \in (0, 1)$. Define x_n via the recurrence

$$x_{n+1} = \frac{3}{4}x_n^2 + \frac{1}{4} \int_0^{|x_n|} f, \quad n \geq 1.$$

Prove that x_n is convergent and find its limit.

2. Suppose (X, d) is a compact metric space with an open cover $\{U_a\}$. Show that for some $\epsilon > 0$, every ball of radius ϵ is fully contained in at least one of the U_a 's.

3. Find

$$\lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} \frac{1}{n^{1+\frac{1}{\log N}}}.$$

Here \log is the natural logarithm (in base e)

4. (a) Give an example of an everywhere differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ whose derivative $f'(x)$ is not continuous.

(b) Show that when $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are functions, and for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $|h| < \delta$ guarantees

$$\left| \frac{f(x+h) - f(x)}{h} - g(x) \right| < \epsilon$$

for all $x \in \mathbb{R}$, then f' exists and is continuous at every $x \in \mathbb{R}$.

5. (a) Give an example of a continuous function on $(0, 1]$ that attains neither a max nor a min on $(0, 1]$.

(b) Show that a uniformly continuous function on $(0, 1]$ must attain either a max or a min on $(0, 1]$.

6. Assume $f : (0, 1)^2 \rightarrow \mathbb{R}$ is continuous and has partial derivative $\frac{\partial f}{\partial x}$ at each point (x, y) satisfying

$$\left| \frac{\partial f}{\partial x}(x, y) \right| \geq 1.$$

Consider the set

$$S_\delta = \{(x, y) \in (0, 1)^2 : |f(x, y)| \leq \delta\}.$$

Prove that the area of S_δ is less than or equal to 4δ for each $\delta > 0$.

7. Prove that there are real-valued continuously differentiable functions $u(x, y)$ and $v(x, y)$ defined on a neighborhood of the point $(1, 2) \in \mathbb{R}^2$ that satisfy the following system of equations,

$$\begin{aligned}xu^2 + yv^2 + xy &= 4 \\ xv^2 + yu^2 - xy &= 1.\end{aligned}$$

8. Consider the upper hemi-ellipsoid surface $\Sigma = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } z \geq 0 \right\}$ for positive constants $a, b, c \in \mathbb{R}$ and define the vector field $\vec{F} = (\partial_y f, -\partial_x f, 2)$ on Σ for some smooth function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$. Evaluate the surface integral $\int_{\Sigma} \vec{F} \cdot \vec{n} \, dS$, where \vec{n} is the upper/outward pointing unit normal field of Σ .

9. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous and suppose that for some $R > 0$, $|f(x, y)| < e^{-\sqrt{x^2+y^2}}$ whenever $\sqrt{x^2 + y^2} \geq R$.

(a) Show that the integral

$$g(s, t) = \int \int_{\mathbb{R}^2} f(x, y) ((x - s)^2 + (y - t)^2) \, dx dy$$

converges for all $(s, t) \in \mathbb{R}^2$

(b) Show that g is continuous on \mathbb{R}^2 .

Tier I ANALYSIS EXAM

August 2019

Try to solve all 9 problems. They each count the same amount. Justify your answers.

1. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Show that the function f has a directional derivative in the direction of any unit vector $\mathbf{v} \in \mathbb{R}^2$ at the origin.

(b) Show that the function f is not continuous at the origin.

2. (a) Prove that if the infinite series

$$(*) \quad \sum_{n=1}^{\infty} |a_{n+1} - a_n| \quad \text{converges for some sequence } \{a_n\} \subset \mathbb{R},$$

then necessarily the sequence $\{a_n\}$ converges as well.

(b) Give an example of a sequence $\{a_n\}$ such that $(*)$ holds while the series

$$\sum_{n=1}^{\infty} a_n \quad \text{diverges.}$$

3. Let $f : [0, 1] \rightarrow \mathbb{R}$ be Riemann integrable and continuous at 0. Show that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x^n) dx = f(0).$$

4. Let

$$\mathbf{F} = \cos(y^2 + z^2)\mathbf{i} + \sin(z^2 + x^2)\mathbf{j} + e^{x^2+y^2}\mathbf{k}$$

be a vector field on \mathbb{R}^3 . Calculate $\int_S \mathbf{F} \cdot d\mathbf{S}$, where the surface S is defined by

$$x^2 + y^2 = e^z \cos z, \quad 0 \leq z \leq \pi/2, \quad \text{and oriented upward.}$$

5. For positive integers n and m suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous and suppose $K \subset \mathbb{R}^n$ is compact. Give a proof that $f(K)$ is compact, that is, give a proof of the fact that the image of a compact set in \mathbb{R}^n under a continuous map is compact.

6. Suppose that $f : (0, \infty) \rightarrow (0, \infty)$ is a differentiable and positive function. Show that for any constant $a > 1$, it must hold that

$$\liminf_{x \rightarrow \infty} \frac{f'(x)}{(f(x))^a} \leq 0.$$

Hint: You might consider an argument that proceeds by contradiction.

7. Prove that the following series

$$\sum_{n=1}^{\infty} \frac{3n^2 + x^4 \cos(nx)}{n^4 + x^2}$$

converges to a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$.

8. Consider the two functions

$$F(x, y, z) := xe^{2y} + ye^z - ze^x$$

and

$$G(x, y, z) := \ln(1 + x + 2y + 3z) + \sin(2x - y + z).$$

(a) Argue that in a neighborhood of $(0, 0, 0)$, the set

$$\{(x, y, z) : F(x, y, z) = 0\} \cap \{(x, y, z) : G(x, y, z) = 0\}$$

can be represented as a continuously differentiable curve parametrized by x .

(b) Find a vector that is tangent to this curve at the origin.

9. Let $\{f_n\}$ be a monotone sequence of continuous functions on $[a, b]$, that is, $f_1(x) \leq f_2(x) \leq f_3(x) \leq \dots$ for all $x \in [a, b]$. Suppose $\{f_n\}$ converges pointwise to a function f which is also continuous on $[a, b]$, as $n \rightarrow \infty$. Show that the convergence is uniform on $[a, b]$.

TIER 1 ANALYSIS EXAM, JANUARY 2020

Write the solution to each of the following problems on a separate, clearly identified page. Each problem is graded on a scale of zero to ten.

Problem 1. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers such that $\lim_{n \rightarrow \infty} a_n = 0$. Show that there exist infinitely many $n \in \mathbb{N}$ with the following property:

$$a_m \leq a_n \text{ for every } m \geq n.$$

Problem 2. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers such that $\lim_{n \rightarrow \infty} a_n = 0$ and

$$|a_n - a_{n+1}| \leq \frac{1}{n^2} \text{ for every } n \in \mathbb{N}.$$

Prove that the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

Problem 3. Denote by X the collection of all sequences $x = \{x_n\}_{n \in \mathbb{N}}$ with the property that $x_n \in [0, 1]$ for every $n \in \mathbb{N}$. Define a metric on X by

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|, \quad x = \{x_n\}_{n \in \mathbb{N}}, y = \{y_n\}_{n \in \mathbb{N}} \in X.$$

Let $f : X \rightarrow \mathbb{R}$ be a uniformly continuous function. Show that f is bounded. (NOTE: Take for granted the fact that d is in fact a metric. The conclusion is not correct if f is just continuous.)

Problem 4. Define a sequence $\{a_n\}_{n \in \mathbb{N}}$ as follows:

$$a_1 = 1, a_2 = \sqrt{2}, a_3 = \sqrt{2\sqrt{3}}, \dots, a_n = \sqrt{2\sqrt{3\sqrt{\dots\sqrt{n}}}}, \quad n \geq 3.$$

Show that the sequence converges in \mathbb{R} .

Problem 5. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a differentiable function such that $f(0, 0) = 0$. Show that the improper integral

$$\iint_{x^2+y^2 \leq 1} \frac{f(x, y)}{(x^2 + y^2)^{4/3}} dx dy$$

converges, that is,

$$\lim_{\varepsilon \downarrow 0} \iint_{\varepsilon \leq x^2+y^2 \leq 1} \frac{f(x, y)}{(x^2 + y^2)^{4/3}} dx dy$$

exists.

Problem 6. Let $f : \mathbb{R} \rightarrow (0, +\infty)$ be a differentiable function such that $f'(x) > f(x)$ for every $x \in \mathbb{R}$.

(1) Show that there exists a constant $k > 0$ such that

$$(0.1) \quad \lim_{x \rightarrow \infty} f(x)e^{-kx} = +\infty.$$

(2) Find the least upper bound of the numbers k for which (0.1) can be proved.

Problem 7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that f' is uniformly continuous and $\lim_{x \rightarrow +\infty} f(x) = 2020$. Does the limit $\lim_{x \rightarrow +\infty} f'(x)$ necessarily exist? (NOTE: Prove if true, provide an example if false.)

Problem 8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(x+1) = f(x)$ for every $x \in \mathbb{R}$. Define functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, as follows:

$$f_1(x) = f(x), f_n(x) = \frac{1}{2}(f_{n-1}(x - 2^{-n}) + f_{n-1}(x + 2^{-n})), \quad x \in \mathbb{R}, n \geq 2.$$

Show that the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly on \mathbb{R} .

Problem 9. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuously differentiable function. Suppose that the Jacobian determinant $\det Df(0,0)$ is equal to zero. Show that for every $\varepsilon > 0$ there exist $M, \delta > 0$ with the following property:

If B_r is the closed disk of radius $r < \delta$ centered at $(0,0)$, then $f(B_r)$ is contained in a rectangle with sides Mr and εr .

Analysis Tier I exam

August 2020

Instructions:

1. Be sure to fully justify all answers.
2. Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.
3. Please assemble your test with the problems in the proper order.
4. Each problem is worth 11 points.

Problem 1. Let $x_0 > 0$ be a fixed real number and consider the sequence

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{4}{x_n} \right), \quad \text{if } n = 0, 1, 2, 3, \dots,$$

- (a) Show that $x_{n+1} \geq 2$, if $n \geq 0$.
- (b) Show that $x_{n+1} \leq x_n$, if $n \geq 1$.
- (c) Show that $x = \lim_{n \rightarrow \infty} x_n$ exists.
- (d) Find x .

Problem 2. Find the value of $\iint_E \mathbf{F} \cdot \mathbf{n} \, dS$ where $\mathbf{F}(x, y, z) = (yz^2, \sin x, x^2)$, E is the upper half of the ellipsoid $\{x^2 + y^2 + 4z^2 = 1, 0 \leq z\}$, and \mathbf{n} is the outward pointing unit normal vector on the ellipsoid.

Problem 3. Find the value of

$$\iint_D \frac{1}{4x+y} \exp\left(\frac{2x+y}{4x+y}\right) \, dx dy$$

where D is the quadrilateral with vertices $(1, -2)$, $(1/2, -1)$, $(1, -3)$, $(2, -6)$.

Problem 4. Find the absolute minimum of the function $f: \mathbb{R}^4 \rightarrow \mathbb{R}$ given by

$$f(x, y, z, w) = x^2y + y^2z + z^2w + w^2x$$

on the set

$$S = \{(x, y, z, w) \in \mathbb{R}^4 : xyzw = 1 \text{ and } x > 0, y > 0, z > 0, w > 0\}.$$

Problem 5. Set $a_0 := 0$ and define for $k \geq 1$

$$a_k = \sqrt{1 + \frac{1}{2} + \dots + \frac{1}{k}}.$$

Assume furthermore that b_k is sequence of positive real numbers such that $\sum_{k=1}^{\infty} b_k^2 < \infty$, and that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous, positive valued function so that

$$f(x) \leq b_k \quad \text{when } a_{k-1} \leq |x| \leq a_k$$

for $k = 1, 2, 3, \dots$. Show that the improper integral $\int_{\mathbb{R}^2} f(x) dx$ exists.

Problem 6. Let f be a continuous function on $[0, 1]$ and twice differentiable on $(0, 1)$ such that $f(0) = f(1) = 0$ and $|f''(x)| < 2$ for all $x \in (0, 1)$.

(a) Show that $f(x) \geq x^2 - x$ for all $x \in [0, 1]$.

(b) Show that

$$\left| \int_0^1 f(x) dx \right| \leq \frac{1}{6}.$$

Problem 7. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a differentiable map (but not necessarily continuously differentiable) with component functions f_1 and f_2 , that is $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ for all $(x_1, x_2) \in \mathbb{R}^2$. Suppose that for all $(x_1, x_2) \in \mathbb{R}^2$, one has

$$\left| \frac{\partial f_1}{\partial x_1}(x_1, x_2) - 2 \right| + \left| \frac{\partial f_1}{\partial x_2}(x_1, x_2) \right| + \left| \frac{\partial f_2}{\partial x_1}(x_1, x_2) \right| + \left| \frac{\partial f_2}{\partial x_2}(x_1, x_2) - 2 \right| \leq \frac{1}{2}$$

Prove that f is one-to-one¹ on \mathbb{R}^2 .

Problem 8. Let I be the interval $[0, 1]$, and let $f : I \rightarrow \mathbb{R}$ be a continuous function such that

$$\int_I f(x)x^n dx = 0 \quad \text{for all } n = 3, 4, 5, \dots$$

Show that $f(x) = 0$ for all $x \in I$.

Problem 9. Let $f_n : [0, 1] \rightarrow [0, 1]$ be a sequence of functions that converge uniformly to a limit function $f : [0, 1] \rightarrow [0, 1]$. Assume that each f_n maps compact sets to compact sets. Is it true that f also maps compact sets to compact sets? Note that we do not assume that the f_n are continuous. Either give a proof, or provide a detailed counterexample.

¹i.e., injective

TIER 1 ANALYSIS EXAM

August 2021

Instructions: There are nine problems, each of equal value. Justify all of your steps, either by direct reasoning or by reference to an appropriate theorem.

1. Let \mathbb{N} be the set of positive integers. Define a distance function $d : \mathbb{N} \times \mathbb{N} \rightarrow [0, \infty)$ such that (\mathbb{N}, d) is a metric space that is not complete. Verify that your choice for d is indeed a metric, and that (\mathbb{N}, d) is not complete.

2. Find all values of x and y minimizing the function $f(x, y) = x/y + y/x$ on the set $x, y > 0, x^2 + 2y^2 = 3$.

3. Let P be the solid parallelepiped in \mathbb{R}^3 with vertices $p_0 = (0, 0, 0)$, $p_1 = (1, 2, 3)$, $p_2 = (2, -1, 5)$, $p_3 = (-1, 7, 4)$, $p_4 = (3, 1, 8)$, $p_5 = (0, 9, 7)$, $p_6 = (1, 6, 9)$, and $p_7 = (2, 8, 12)$. (Note: If the p_i are considered as vectors, then $p_4 = p_1 + p_2$, $p_5 = p_1 + p_3$, $p_6 = p_2 + p_3$, and $p_7 = p_1 + p_2 + p_3$.) Evaluate

$$\int \int \int_P (-x + 3y + z) \, dx \, dy \, dz.$$

4. Let E be the square-based pyramid in \mathbb{R}^3 with top vertex $(1, 2, 5)$ and base $\{(x, y, 0) : 0 \leq x \leq 3, 0 \leq y \leq 3\}$, and let S_1, S_2, S_3, S_4 be the four triangular sides of E . Define the vector field $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (3x - y + 4z, x + 5y - 2z, x^2 + y^2 - z).$$

Find

$$\sum_{j=1}^4 \int \int_{S_j} \mathbf{F} \cdot \mathbf{n} \, dA,$$

where \mathbf{n} is chosen to be the unit normal vector to S_j with a positive component in the z direction, and dA indicates that the integral is with respect to surface area on S_j .

5. The improper integral $\int_0^\infty g(x) \, dx$ of a continuous function g is defined as $\lim_{R \rightarrow \infty} \int_0^R g(x) \, dx$ when this limit exists. Let f be continuous on \mathbb{R}^2 , and suppose that $\int_0^\infty f(x, y) \, dy$ exists for every $x \in [0, 1]$. Assume there is a positive constant C such that

$$\left| \int_z^\infty f(x, y) \, dy \right| \leq \frac{C}{\log(2+z)}, \quad \text{for } z > 0 \text{ and } 0 \leq x \leq 1.$$

Show that $\int_0^1 \left[\int_0^\infty f(x, y) \, dy \right] dx = \int_0^\infty \left[\int_0^1 f(x, y) \, dx \right] dy$.

6. Assume $a_1 \in (0, 1)$ and

$$a_{n+1} = a_n^3 - a_n^2 + 1, \text{ for } n = 1, 2, 3, \dots$$

- (a) Prove that $\{a_n\}_{n=1}^{\infty}$ converges and find its limit.
(b) For $b_n = a_1 a_2 \cdots a_n$, prove that $\{b_n\}_{n=1}^{\infty}$ converges and find its limit.

7. Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence of continuous functions defined on $[0, 1] \times [0, 1]$, and let $F_n(x, y) = \int_y^1 \left[\int_x^1 s^{-1/2} t^{-1/3} f_n(s, t) ds \right] dt$.

(a) Show that, for each n , $F_n(x, y)$ is well-defined (possibly as an iterated improper integral) for $(x, y) \in [0, 1] \times [0, 1]$. (Recall that the improper integral $\int_0^1 g(u) du$ of a continuous function g on $(0, 1]$ is defined as $\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 g(u) du$ when this limit exists.)

(b) Show that the sequence $\{F_n\}_{n=1}^{\infty}$ has a subsequence $\{F_{n_j}\}_{j=1}^{\infty}$ that converges uniformly on $[0, 1] \times [0, 1]$ to a continuous limit F .

8. We let $\log x$ be the natural logarithm (in base e). Is the series

$$\sum_{n \geq 100} \frac{1}{(\log n)^{\log \log n}}$$

convergent or divergent? Justify your answer.

9. Suppose $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous, and for each $(x, y) \in \mathbb{R}^2$, $z \mapsto F(x, y, z)$ is a strictly increasing function of z . Suppose that $F(x_0, y_0, z_0) = 0$.

(a) Prove that there exists an open neighborhood U of (x_0, y_0) in \mathbb{R}^2 such that there is a unique function $g : U \rightarrow \mathbb{R}$ with $F(x, y, g(x, y)) = 0$ for all $(x, y) \in U$.

(b) Show that g is continuous on U .

TIER 1 ANALYSIS EXAMINATION
JANUARY 4, 2022

The complete solution to each of the problems below is worth 10 points, so 90 is the maximum score. Please write your solutions on separate sheets, use only one side of each sheet, and make sure each page is labeled with a problem number.

- (1) Define continuous functions $f_n : [0, 1] \rightarrow \mathbb{R}$ by

$$f_n(x) = \frac{1 + x^n}{1 + 2^{-n}}, \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

Show that the sequence $(f_n)_{n \in \mathbb{N}}$ is not equicontinuous on $[0, 1]$.

- (2) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that

$$\sum_{n=1}^{\infty} |a_n - a_{n+1}| < +\infty.$$

Show that $(a_n)_{n \in \mathbb{N}}$ is a convergent sequence.

- (3) Let $f : \mathbb{R} \rightarrow [0, +\infty)$ be a differentiable function such that both f and $-f'$ are nonincreasing on \mathbb{R} . Prove that

$$\lim_{x \rightarrow +\infty} f'(x) = 0.$$

- (4) Let $G \subset \mathbb{R}^5$ be the set of vectors $A = (a_0, a_1, a_2, a_3, a_4)$ with the property that the quintic polynomial

$$P_A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + x^5$$

has five distinct real roots. Prove that G is an open set.

- (5) Does the improper integral

$$\int_0^{\infty} \cos(x^{2/3}) dx$$

converge? Justify your answer.

- (6) Let t_0 be an arbitrary real number. Define a sequence $(t_n)_{n \in \mathbb{N}}$ by setting $t_n = \sin(\cos(t_{n-1}))$ for $n \geq 1$. Prove that this sequence converges and that the limit does not depend of t_0 .

- (7) Suppose that $(a_n)_{n \in \mathbb{N}}$ is an unbounded, increasing sequence of positive numbers. Show that the series

$$\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{a_n}$$

diverges.

- (8) Suppose that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous, compactly supported function. Define a new function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$g(x) = \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy, \quad x \in \mathbb{R}^2.$$

Prove that the improper integral does in fact converge and that the function g is continuous. (Here $x = (x_1, x_2)$, $y = (y_1, y_2)$, $|x| = \sqrt{x_1^2 + x_2^2}$, and $dy = dy_1 dy_2$. Convergence of the improper integral means that the Riemann integral

$$\int_{\varepsilon < |x-y| < 1/\varepsilon} \frac{f(y)}{|x-y|} dy$$

has a limit as $\varepsilon \downarrow 0$.)

- (9) Let $F_1(x, y, z) = 6yz$, $F_2(x, y, z) = 2xz$, $F_3(x, y, z) = 4xy$, and let $\alpha, \gamma : [-\pi, \pi] \rightarrow \mathbb{R}^3$ be defined by

$$\alpha(t) = (\cos(t), \sin(t), 0),$$

$$\gamma(t) = (\cos(t), \sin(t), 4 + (\sin(t))(\cos(t^3))).$$

- (a) Apply Stokes' Theorem on the surface

$$S = \{(\cos(t), \sin(t), z) : -\pi \leq t \leq \pi, 0 \leq z \leq 4 + (\sin(t))(\cos(t^3))\}$$

to express

$$\int_{\gamma} (F_1 dx + F_2 dy + F_3 dz)$$

in terms of

$$\int_{\alpha} (F_1 dx + F_2 dy + F_3 dz).$$

- (b) Use (a) to evaluate the first integral.