Analysis Tier I exam

August 2020

Instructions:

- 1. Be sure to fully justify all answers.
- 2. Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.
- 3. Please assemble your test with the problems in the proper order.
- 4. Each problem is worth 11 points.

Problem 1. Let $x_0 > 0$ be a fixed real number and consider the sequence

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{4}{x_n} \right), \text{ if } n = 0, 1, 2, 3, \dots,$$

- (a) Show that $x_{n+1} \geq 2$, if $n \geq 0$.
- (b) Show that $x_{n+1} \leq x_n$, if $n \geq 1$.
- (c) Show that $x = \lim_{n \to \infty} x_n$ exists.
- (d) Find x.

Problem 2. Find the value of $\iint_E \mathbf{F} \cdot \mathbf{n} \ dS$ where $\mathbf{F}(x,y,z) = (yz^2, \sin x, x^2)$, E is the upper half of the ellipsoid $\{x^2 + y^2 + 4z^2 = 1, \ 0 \le z\}$, and \mathbf{n} is the outward pointing unit normal vector on the ellipsoid.

Problem 3. Find the value of

$$\iint_D \frac{1}{4x+y} \exp\left(\frac{2x+y}{4x+y}\right) dxdy$$

where D is the quadrilateral with vertices (1, -2), (1/2, -1), (1, -3), (2, -6).

Problem 4. Find the absolute minimum of the function $f: \mathbb{R}^4 \to \mathbb{R}$ given by

$$f(x, y, z, w) = x^2y + y^2z + z^2w + w^2x$$

on the set

$$S = \{(x, y, z, w) \in \mathbb{R}^4 : xyzw = 1 \quad and \quad x > 0, \ y > 0, \ z > 0, \ w > 0\} \ .$$

Problem 5. Set $a_0 := 0$ and define for $k \ge 1$

$$a_k = \sqrt{1 + \frac{1}{2} + \ldots + \frac{1}{k}}$$
.

Assume furthermore that b_k is sequence of positive real numbers such that $\sum_{k=1}^{\infty} b_k^2 < \infty$, and that $f: \mathbb{R}^2 \to \mathbb{R}$ is a continuous, positive valued function so that

$$f(x) \le b_k$$
 when $a_{k-1} \le |x| \le a_k$

for $k = 1, 2, 3, \ldots$ Show that the improper integral $\int_{\mathbb{R}^2} f(x) dx$ exists.

Problem 6. Let f be a continuous function on [0,1] and twice differentiable on (0,1) such that f(0) = f(1) = 0 and |f''(x)| < 2 for all $x \in (0,1)$.

- (a) Show that $f(x) \ge x^2 x$ for all $x \in [0, 1]$.
- (b) Show that

$$\left| \int_0^1 f(x) \ dx \right| \le \frac{1}{6} \ .$$

Problem 7. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map (but not necessarily continuously differentiable) with component functions f_1 and f_2 , that is $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ for all $(x_1, x_2) \in \mathbb{R}^2$. Suppose that for all $(x_1, x_2) \in \mathbb{R}^2$, one has

$$\left| \frac{\partial f_1}{\partial x_1}(x_1, x_2) - 2 \right| + \left| \frac{\partial f_1}{\partial x_2}(x_1, x_2) \right| + \left| \frac{\partial f_2}{\partial x_1}(x_1, x_2) \right| + \left| \frac{\partial f_2}{\partial x_2}(x_1, x_2) - 2 \right| \le \frac{1}{2}$$

Prove that f is one-to-one¹ on \mathbb{R}^2 .

Problem 8. Let I be the interval [0,1], and let $f: I \to \mathbb{R}$ be a continuous function such that

$$\int_{I} f(x)x^{n} dx = 0 \text{ for all } n = 3, 4, 5 \dots$$

Show that f(x) = 0 for all $x \in I$.

Problem 9. Let $f_n:[0,1] \to [0,1]$ be a sequence of functions that converge uniformly to a limit function $f:[0,1] \to [0,1]$. Assume that each f_n maps compact sets to compact sets. Is it true that f also maps compact sets to compact sets? Note that we do not assume that the f_n are continuous. Either give a proof, or provide a detailed counterexample.

 $^{^{1}}$ i.e., injective