Tier 1 Analysis Exam

January 2021

Work all nine problems. They all count equally. Show computations and justify your answers; a correct answer without a correct proof earns little credit. Write a solution of each problem on a separate page. You have 4 hours.

Notation: For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, denote the partial derivative in the i-th coordinate direction by $D_i f$. The partial derivative of $D_i f$ in the j-th coordinate direction is likewise denoted by $D_{ij} f := D_j D_i f$. The expression $:=$ is used to indicate a definition.

1. Let $B(K)$ denote the set of bounded functions $f\colon K \to \mathbb{R}$, where $K \subset \mathbb{R}$ is compact.

   (a) For $f, g \in B(K)$, define
   $$d(f,g) := \sup_{x\in K} |f(x) - g(x)|.$$
   Show that $d\colon B(K) \times B(K) \to \mathbb{R}$ defines a metric on $B(K)$.

   (b) Show that the set $C(K)$ of continuous functions $f\colon K \to \mathbb{R}$ is a closed subset of $B(K)$ in the topology given by this metric.

2. Let $a_{n,m} \in [0,1]$ for all positive integers n and m. Suppose that for each n, we have $\lim_{m\to\infty} a_{n,m} = n/2^n$. For each of the following inequalities, prove that it must hold or prove (with a counterexample) that it need not hold:

   (a) $\displaystyle \liminf_{m\to\infty} \sum_{n=1}^{\infty} \frac{a_{n,m}}{n} \geq 1;$

   (b) $\displaystyle \limsup_{m\to\infty} \sum_{n=1}^{\infty} \frac{a_{n,m}}{n} \leq 1.$

3. Let $a_k \geq 0$ for all nonnegative integers k. Suppose that $\sum_{k=0}^{\infty} a_k < \infty$. Define $f(x) := \sum_{k=0}^{\infty} a_k x^k$ for $x \in [0,1]$. Do not assume that $b := \sum_{k=0}^{\infty} k\, a_k$ is finite.

   (a) Show that (the left-hand derivative) $f'(1) = b$.

   (b) Show that $\lim_{x\to 1^-} f'(x) = b$.

4. Let $a_k \geq 0$ for all nonnegative integers k. Suppose that $\sum_{k=0}^{\infty} a_k = 1$, $\sum_{k=0}^{\infty} k\, a_k = 1$, and $c := \sum_{k=0}^{\infty} k(k-1)\, a_k$ is finite. Define $f(x) := \sum_{k=0}^{\infty} a_k x^k$ for $x \in [0,1]$. You may assume without proof that $\lim_{x\to 1^-} f'(x) = 1$ and $\lim_{x\to 1^-} f''(x) = c$. (These both follow from problem 3(b).) Define
   $$g(x) := \frac{1}{1 - f(x)} - \frac{1}{1-x}$$
   for $x \in [0,1)$. Show that $\lim_{x\to 1^-} g(x) = c/2$.

5. Consider a function $f\colon \mathbb{R}^2 \to \mathbb{R}$. Suppose that $D_1 f$ exists at $(0,0)$. Suppose also that $D_2 f$ exists in a neighborhood of $(0,0)$ and is continuous at $(0,0)$. Prove that f is differentiable at $(0,0)$.

6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable in a neighborhood of $(0,0)$, with $D_2 f(0,0) = 0$ and $D_{22} f(0,0) > 0$.

   (a) Prove that there are $\epsilon, \delta > 0$ such that for each $x \in (-\epsilon, \epsilon)$, the formula
   $$g(x) := \min\{f(x,y) : |y| \le \delta\}$$
   defines a differentiable function $g\colon (-\epsilon, \epsilon) \to \mathbb{R}$.

   (b) Find a formula for $g'(0)$ in terms of f and prove it.

7. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) := x^4 + y^4 - 4xy$.

   (a) Identify and classify all critical points of f on $\mathbb{R}^2$.

   (b) Determine the minimum and maximum values of f on the curve $x^4 + y^4 = 32$.

8. Let $\mathbb{Q} \subset \mathbb{R}$ denote the rationals. Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by
   $$f(x,y) := \begin{cases} x & \text{if } (x,y) \in \mathbb{Q} \times \mathbb{Q}, \\ y & \text{otherwise} \end{cases}$$
   is not Riemann integrable on the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$.

9. Let $S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ with } y \ge 0 \text{ and } z \ge 0\}$ be the surface consisting of a quarter of the unit sphere in $\mathbb{R}^3$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be infinitely differentiable. Define the vector field $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{F}(x,y,z) := (f(z) + x^2 y, xy^2 + 1, xyz)$. Evaluate the surface integral $\int_S \mathbf{F} \cdot \hat{n}\, dS$, where $\hat{n}$ is the unit normal vector field of S pointing away from the origin.