TIER 1 ANALYSIS EXAM

August 2021

Instructions: There are nine problems, each of equal value. Justify all of your steps, either by direct reasoning or by reference to an appropriate theorem.

1. Let $\mathbb{N}$ be the set of positive integers. Define a distance function $d : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ such that $(\mathbb{N}, d)$ is a metric space that is not complete. Verify that your choice for d is indeed a metric, and that $(\mathbb{N}, d)$ is not complete.

2. Find all values of x and y minimizing the function $f(x, y) = x/y + y/x$ on the set $x, y > 0$, $x^2 + 2y^2 = 3$.

3. Let P be the solid parallelepiped in $\mathbb{R}^3$ with vertices $p_0 = (0, 0, 0)$, $p_1 = (1, 2, 3)$, $p_2 = (2, -1, 5)$, $p_3 = (-1, 7, 4)$, $p_4 = (3, 1, 8)$, $p_5 = (0, 9, 7)$, $p_6 = (1, 6, 9)$, and $p_7 = (2, 8, 12)$. (Note: If the p_i are considered as vectors, then $p_4 = p_1 + p_2$, $p_5 = p_1 + p_3$, $p_6 = p_2 + p_3$, and $p_7 = p_1 + p_2 + p_3$.) Evaluate

$$\int\int\int_P (-x + 3y + z) \quad dx\ dy\ dz.$$

4. Let E be the square-based pyramid in $\mathbb{R}^3$ with top vertex $(1, 2, 5)$ and base $\{(x, y, 0) : 0 \leq x \leq 3, 0 \leq y \leq 3\}$, and let S_1, S_2, S_3, S_4 be the four triangular sides of E. Define the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (3x - y + 4z, x + 5y - 2z, x^2 + y^2 - z).$$

Find

$$\sum_{j=1}^{4} \int\int_{S_j} \mathbf{F} \cdot \mathbf{n}\, dA,$$

where $\mathbf{n}$ is chosen to be the unit normal vector to S_j with a positive component in the z direction, and dA indicates that the integral is with respect to surface area on S_j.

5. The improper integral $\int_0^\infty g(x)\, dx$ of a continuous function g is defined as $\lim_{R\to\infty} \int_0^R g(x)\, dx$ when this limit exists. Let f be continuous on $\mathbb{R}^2$, and suppose that $\int_0^\infty f(x, y)\, dy$ exists for every $x \in [0, 1]$. Assume there is a positive constant C such that

$$\left| \int_z^\infty f(x, y)\, dy \right| \leq \frac{C}{\log(2 + z)}, \quad \text{for } z > 0 \text{ and } 0 \leq x \leq 1.$$

Show that $\int_0^1 \left[\int_0^\infty f(x, y)\, dy \right] dx = \int_0^\infty \left[\int_0^1 f(x, y)\, dx \right] dy.$

6. Assume $a_1 \in (0,1)$ and

$$a_{n+1} = a_n^3 - a_n^2 + 1, \text{for } \ n = 1, 2, 3, \ldots.$$

(a) Prove that $\{a_n\}_{n=1}^{\infty}$ converges and find its limit.
(b) For $b_n = a_1 a_2 \cdots a_n$, prove that $\{b_n\}_{n=1}^{\infty}$ converges and find its limit.

7. Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence of continous functions defined on $[0,1] \times [0,1]$, and let $F_n(x,y) = \int_y^1 \left[\int_x^1 s^{-1/2} t^{-1/3} f_n(s,t) ds\right] dt$.

(a) Show that, for each n, $F_n(x,y)$ is well-defined (possibly as an iterated improper integral) for $(x,y) \in [0,1] \times [0,1]$. (Recall that the improper integral $\int_0^1 g(u)\ du$ of a continuous function g on $(0,1]$ is defined as $\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 g(u)\ du$ when this limit exists.)

(b) Show that the sequence $\{F_n\}_{n=1}^{\infty}$ has a subsequence $\{F_{n_j}\}_{j=1}^{\infty}$ that converges uniformly on $[0,1] \times [0,1]$ to a continuous limit F.

8. We let $\log x$ be the natural logarithm (in base e). Is the series

$$\sum_{n \geq 100} \frac{1}{(\log n)^{\log \log n}}$$

convergent or divergent? Justify your answer.

9. Suppose $F : \mathbb{R}^3 \to \mathbb{R}$ is continuous, and for each $(x,y) \in \mathbb{R}^2$, $z \mapsto F(x,y,z)$ is a strictly increasing function of z. Suppose that $F(x_0, y_0, z_0) = 0$.

(a) Prove that there exists an open neighborhood U of (x_0, y_0) in $\mathbb{R}^2$ such that there is a unique function $g : U \to \mathbb{R}$ with $F(x, y, g(x,y)) = 0$ for all $(x,y) \in U$.

(b) Show that g is continous on U.