TIER 1 ANALYSIS EXAM, AUGUST 2016

Directions: Be sure to use **separate** pieces of paper for different solutions. This exam consists of nine questions and each counts equally. Credit may be given for partial solutions.

(1) Let $f : [0,1] \to \mathbb{R}$ be an nondecreasing function, and let D be the set of $x \in [0,1]$ such that f is not continuous at x. Is the set D necessarily compact? Fully justify your answer.

(2) Show that there exist a real number $\varepsilon > 0$ and a differentiable function $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$e^{x^2+f(x)} \;=\; 1 - \sin(x + f(x)).$$

(3) Prove that the function f defined by

$$f(x) := \sum_{n=0}^{\infty} \frac{\cos(n^2 x)}{2^{nx}}$$

is continuous on the interval $(0, \infty)$.

(4) Using only the definitions of continuity and (sequential) compactness, prove that if $K \subset \mathbb{R}$ is (sequentially) compact and $f : K \to \mathbb{R}$ is continuous, then f is uniformly continuous, that is, for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

(5) Show that if $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, then the set of limit of points of $\{x_n\}$ is connected, that is, either empty, a single point, or an interval.

(6) Let a and b be positive numbers, and let Γ be the closed curve in $\mathbb{R}^3$ that is the intersection of the surface $\{(x,y,z) : z = b \cdot x \cdot y\}$ and the cylinder $\{(x,y,z) : x^2 + y^2 = a^2\}$. Let r be a parametrization of Γ so that the curve is oriented counter-clockwise when looking down upon it from high up on the z-axis. Compute

$$\int_{\Gamma} F \cdot dr.$$

where F is the vector valued function defined by $F(x,y,z) = (y, z, x)$.

(7) Let $\Omega = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, and define $f : \Omega \to \mathbb{R}$ by

$$f(x, y) \;=\; \frac{2 + \sqrt{(1 + x)^2 + y^2} + \sqrt{(1 - x)^2 + y^2}}{\sqrt{y}}.$$

Show that f has achieves its minimum value on Ω at a unique point $(x_0, y_0) \in \Omega$ and find (x_0, y_0).

(8) Suppose that $(a_n)_{n=1}^{\infty}$ is a bounded sequence of positive numbers. Show that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} \;=\; 0$$

if and only if

$$\lim_{n\to\infty} \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \;=\; 0.$$

(9) Define $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d(x, y) \;=\; \frac{\|x - y\|}{\|x\|^2 + \|y\|^2 + 1}$$

where $\|x\|^2 = x_1^2 + \cdots + x_n^2$. Let $A \subset \mathbb{R}^n$ be such that there exists $\epsilon > 0$ so that if $a, b \in A$ with $a \neq b$, then $d(a, b) \geq \epsilon$. Show that A is finite.