TIER I ANALYSIS EXAM, JANUARY 2016

Solve all nine problems. They all count equally. Show all computations.

1. Let $a > 0$ and let x_n be a sequence of real numbers. Assume the sequence

$$y_n = \frac{x_1 + x_2 + \ldots + x_n}{n^a}$$

is bounded. Show that for each $b > a$, the series

$$\sum_{n=1}^{\infty} \frac{x_n}{n^b}$$

is convergent.

2. (a) Show that for each integer $n \geq 1$ there exists exactly one $x > 0$ such that

$$\frac{1}{\sqrt{nx+1}} + \frac{1}{\sqrt{nx+2}} + \ldots + \frac{1}{\sqrt{nx+n}} = \sqrt{n}.$$

(b) Call x_n the solution from (a). Find

$$\lim_{n \to \infty} x_n.$$

3. Let (X, d) be a compact metric space and let ρ be another metric on X such that

$$\rho(x, x') \leq d(x, x'), \text{ for all } x, x' \in X.$$

Show that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\rho(x, x') < \delta \implies d(x, x') < \epsilon.$$

4. Prove that for each $x \in \mathbb{R}$ there is a choice of signs $s_n \in \{-1, 1\}$ such that the series

$$\sum_{n=1}^{\infty} \frac{s_n}{\sqrt{n}}$$

converges to x.

5. Assume the function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the property

$$f(x+t, y+s) \geq f(x, y) - s^2 - t^2,$$

for each $(x, y) \in \mathbb{R}^2$ and each $(s, t) \in \mathbb{R}^2$. Prove that f must be constant.

6. Assume $f : [0, 1] \to \mathbb{R}$ is continuous and $f(0) = 2016$. Find

$$\lim_{n \to \infty} \int_0^1 f(x^n) dx.$$

7. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be two differentiable functions with $f(x, y, z) = g(xy, yz)$ and suppose that $g(u, v)$ satisfies

$$g(2,6) = 2, \ \frac{\partial g}{\partial u}(2,6) = -1, \quad \text{and} \ \frac{\partial g}{\partial v}(2,6) = 3.$$

Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 2\}$ admits a tangent plane at the point $(1, 2, 3)$, and find an equation for it.

8. Let $\mathcal{C}$ be the collection of all positively oriented (i.e. counter-clockwise) simple closed curves C in the plane. Find

$$\sup\{\int_C (y^3 - y)dx - 2x^3 dy : C \in \mathcal{C}\}.$$

Is the supremum attained?

9. Let

$$H = \{(x, y, z) \,|\, z > 0 \text{ and } x^2 + y^2 + z^2 = R^2\}$$

be the upper hemisphere of the sphere of radius R centered at the origin in $\mathbb{R}^3$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$F(x, y, z) = \left(x^2 \sin\left(y^2 - z^3\right), xy^4 z + y, e^{-x^2-y^2} + yz\right)$$

Find $\displaystyle\int_H F \cdot \hat{n} \, dS$ where $\hat{n}$ is the outward pointing unit surface normal and dS is the area element.