Analysis Tier I Exam
August 2015

- **Be sure to fully justify all answers.**
- **Scoring:** Each problem is worth 10 points.
- **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
- Please be sure that you assemble your test with the problems presented in correct order.

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1. Let $f(x)$ be a continuous function on $(0,1]$ and

$$\liminf_{x\to 0^+} f(x) = \alpha, \quad \limsup_{x\to 0^+} f(x) = \beta.$$

Prove that for any $\xi \in [\alpha, \beta]$, there exist $\{x_n \in (0,1] \mid n = 1, 2, \cdots\}$ such that

$$\lim_{n\to\infty} f(x_n) = \xi.$$

2. Let $f(x)$ be a function which is defined and is continuously differentiable on an open interval containing the closed interval [a,b], and let

$$f^{-1}(0) = \{x \in [a,b] \mid f(x) = 0\}.$$

Assume that $f^{-1}(0) \neq \emptyset$, and for any $x \in f^{-1}(0)$, $f'(x) \neq 0$. **Prove** the following assertions:

(a) $f^{-1}(0)$ is a finite set;

(b) Let p be the number of points in $f^{-1}(0)$ such that $f'(x) > 0$, and q be the number of points in $f^{-1}(0)$ such that $f'(x) < 0$. Then

$$|p - q| \leq 1.$$

3. Let $\sum_{n=1}^{\infty} a_n$ be a convergent positive term series ($a_n \geq 0$ for all n). **Show** that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. **Is the converse true?**

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with f' uniformly continuous. Suppose $\lim_{x\to\infty} f(x) = L$ for some L. Does $\lim_{x\to\infty} f'(x)$ exist?

5. Let $E \subset \mathbb{R}$ be a set with the property that any countable family of closed sets that cover E contains a finite subcollection which covers E. **Show** that E must consist of finitely many points.

6. Suppose that a function $f(x)$ is defined as the sum of a series:

$$f(x) = 1 - \frac{1}{(2!)^2}(2015x)^2 + \frac{1}{(4!)^2}(2015x)^4 - \frac{1}{(6!)^2}(2015x)^6 + \dots$$
$$= \sum_{k=0}^{\infty}(-1)^k \frac{1}{((2k)!)^2}(2015x)^{2k}.$$

Evaluate

$$\int_0^\infty e^{-x} f(x)\, dx.$$

7. **Find** the volume of the solid S in $\mathbb{R}^3$, which is the intersection of two cylinders $C_1 = \{(x, y, z) \in \mathbb{R}^3;\ y^2 + z^2 \le 1\}$ and $C_2 = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + z^2 \le 1\}$.

8. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Suppose that f has the property that for any compact set $K \subset \mathbb{R}^m$, the set $f^{-1}(K) \subset \mathbb{R}^n$ is bounded. **Prove** that $f(\mathbb{R}^n)$ is a closed subset of $\mathbb{R}^m$, **or give** a counterexample to this claim.

9. Let $F : \mathbb{R}^2 \to \mathbb{R}$ have continuous second-order partial derivatives. **Find all points** where the condition in the implicit function theorem is satisfied so that $F(x - y, y - z) = 0$ defines an implicit function $z = z(x, y)$, and **derive** explicit formulas, in terms of partial derivatives of F, for

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x \partial y}.$$

10. Suppose that a monotone sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a continuous function F on some closed interval $[a, b]$. **Prove** that the convergence is uniform.

Note: In this problem by a monotone sequence of functions we mean a sequence f_n such that either $f_n(x) \le f_{n+1}(x)$ for all n and all $x \in [a, b]$, or $f_n(x) \ge f_{n+1}(x)$ for all n and all $x \in [a, b]$.