Tier 1 Analysis Exam

JANUARY 5, 2015

You have 4 hours to work these 10 problems. Each is worth 10 points.

- *Start each answer on on a clean sheet of paper*
- *Use only one side of each sheet*
- *Circle the prob. number in the upper-right corner of each sheet*
- *Fully justify all answers.*
- *Put your answers in the correct order before submitting them.*

0.1. An open set $U \subset \mathbf{R}^n$ contains the closed origin-centered unit ball $B = B(\mathbf{0}, 1)$. If a C^1 mapping $f \colon U \to \mathbf{R}^n$ with rank n obeys $\|f(x) - x\| < 1/2$ for all $x \in U$, show that

a) $\|f\|^2$ must attain a minimum in the interior of B.

b) $f(p) = \mathbf{0}$ for some $p \in B$.

0.2. Suppose $f, g \colon \mathbf{R} \to \mathbf{R}$, are functions that obey

$$f(x+h) = f(x) + g(x)h + a(x,h)$$

for **all** $x, h \in \mathbf{R}$, with $|a(x,h)| \leq Ch^3$ for some constant C.
Show that f is affine (i.e., $f(x) = mx + b$ for some $m, b \in \mathbf{R}$).

0.3. Suppose f is differentiable on an open interval containing $[-1,1]$. Do **not** assume continuity of f'.

a) Supposing $f'(-1)f'(1) < 0$ show that $f'(x) = 0$ for some $x \in (-1,1)$.

b) Supposing that $f'(-1) < L < f'(1)$ for some $L \in \mathbf{R}$, show that $f'(x) = L$ for some $x \in (-1,1)$.

0.4. Suppose (X,d) is a complete metric space. Show that if every continuous function on a subset $U \subset X$ attains a minimum, then U is closed.

0.5. Define the distance from a point p in a metric space (X,d) to a *subset* $Y \subset X$ by

$$d(p,Y) := \inf\{d(x,y) \colon y \in Y\}$$

For any $\varepsilon > 0$, define

$$Y_\varepsilon = \{x \in X \colon d(x,Y) \leq \varepsilon\}$$

Finally, given any two **bounded** sets $A, B \subset X$, define

$$d_S(A,B) = \inf\{\varepsilon > 0\colon A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\}$$

(a) Show that d_S yields a metric on the set of **closed bounded** subsets of X.

(b) Show that d_S fails to do so on the set of **bounded** subsets of X.

0.6. Determine whether the series converges or not.

$$\sum_{j=1}^{\infty} \left(e^{(-1)^j \sin(1/j)} - 1 \right)$$

0.7. Let B_r denote the ball $|x| \leq r$ in $\mathbf{R}^3$, and write dS_r for the area element on its boundary ∂B_r.

The electric field associated with a uniform charge distribution on ∂B_R may be expressed as

$$E(x) = C \int_{\partial B_R} \nabla_x |x-y|^{-1}\, dS_y,$$

a) Show that for any $r < R$, the electric flux $\int_{\partial B_r} E(x) \cdot \nu\, dS_x$ through ∂B_r equals zero.

b) Show that $E(x) \equiv 0$ for $|x| < R$ ("a conducting spherical shell shields its interior from outside electrical effects").

0.8. Let Q be a bounded closed rectangle in $\mathbf{R}^n$, and suppose we have functions $f, g\colon Q \to \mathbf{R}$ that, for some $K > 0$, satisfy

$$|f(x) - f(y)| \leq K\, |g(x) - g(y)|$$

and all $x, y \in Q$. Prove that if g is Riemann integrable, then so is f. Deduce further that integrability of f implies that of $|f|$.

0.9. Suppose $f\colon U \to \mathbf{R}$ is a differentiable function defined on an open set $U \supset [0,1]^2$. Assuming $f(0,0) = 3$ and $f(1,1) = 1$, prove that for $|\nabla f| \geq \sqrt{2}$ somewhere in U.

0.10. Consider this quadratic system in $\mathbf{R}^4$:

$$a^2 + b^2 - c^2 - d^2 = 0$$
$$ac + bd = 0$$

Show the system can be solved for (a, c) in terms of (b, d) (or vice-versa) near any solution $(a_0, b_0, c_0, d_0) \neq (0,0,0,0)$. (You need not find explicit solutions here.)