Tier 1 Analysis Exam

January 6, 2014

Each problem below is worth 10 points. Answer each one on a new sheet of paper, writing the problem number on every sheet. Use only one side of each sheet, and fully justify all answers. Put your answers in the correct order when you turn them in. You have 4 hours.

0.1. Suppose a metric space (X, d) has this property: *Given any* $\varepsilon > 0$, *there is a non-empty* finite *subset* $X_\varepsilon \subset X$ *such that for every* $x \in X$, *we have*

$$\inf\{d(x,p) \colon p \in X_\varepsilon\} \leq \varepsilon$$

a) Show that in this case, every sequence in X has a Cauchy subsequence.

b) Give an example showing that (a) fails if we don't require the X_ε's to be finite.

0.2. For $p, q \in \mathbf{R}^3$, let $|p|$ and $p \times q$ respectively denote the euclidean norm of p, and the cross-product of p and q. Define $d : \mathbf{R}^3 \times \mathbf{R}^3 \to [0, \infty)$ by

$$d(p,q) = \begin{cases} |p| + |q|, & p \times q \neq 0 \\ |p - q|, & p \times q = 0 \end{cases}$$

a) Show that d is a metric on $\mathbf{R}^3$.

b) Show that the closed unit d-ball centered at $(0,0,0)$ is **not** d-compact.

c) Show that the closed unit d-ball centered at $(1,1,1)$ **is** d-compact.

0.3. Assume $f, \omega : \mathbf{R} \to \mathbf{R}$ are functions, with $\omega(0) = 0$. Assume too that for some $\alpha > 1$, we have

$$f(b) \leq f(a) + \omega(|b-a|)^\alpha \quad \text{for all } a, b \in \mathbf{R} \tag{1}$$

a) Show that when ω is differentiable at $x = 0$, our assumptions make f infinitely differentiable at every point.

b) Give an example showing that when $\alpha > 1$ but ω is merely continuous, our assumptions do not force differentiability of f at all points.

0.4. Show that every sequence in $\mathbf{R}$ has a weakly monotonic (i.e. non-increasing, or non-decreasing) subsequence.

0.5. Show that the series converges, but not absolutely:

$$\sum_{n=1}^{\infty}\left(\exp\left(\frac{(-1)^n}{n}\right)-1\right)$$

0.6. Consider this integral:

$$\int_0^{\infty} \sin(x^p)\, dx$$

a) Does it converge when $p = 1$?

b) Does it converge when $p < 0$?

c) Does it converge when $p > 1$?

0.7. Suppose $f : [0,\infty) \to [0,\infty)$ is a continuous bijection and consider the series

$$\sum_{n=1}^{\infty} \frac{n f(x^2)}{1+n^3 f(x^2)^2}$$

a) Show that the series converges pointwise for all $x \in \mathbf{R}$.

b) Show that it converges *uniformly* on $[\varepsilon,\infty)$ when $\varepsilon > 0$.

c) Show that it does *not* converge uniformly on $\mathbf{R}$.

0.8. Let S denote the upper hemisphere of radius $r > 0$ centered at $\mathbf{0} \in \mathbf{R}^3$, i.e.,

$$S = \{(x,y,z) \,|\, x^2+y^2+z^2 = r^2 \text{ and } z \geq 0\}$$

and suppose $F : \mathbf{R}^3 \to \mathbf{R}^3$ is the vector field given by

$$F(x,y,z) \quad = \quad \begin{pmatrix} x\,y^2 \tanh(x^2+z) \\ x + y^4 \sin(z)\, e^{-x^2} \\ x^2(x^3+3)\, y\, e^{-x^2-y^2-z^2} \end{pmatrix}.$$

Compute

$$\int_S \operatorname{curl}(F) \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is the upward pointing unit surface normal, and dS is the area element on S.

0.9. Consider this system of equations in the variables u, v, s, t:

$$(uv)^4 + (u+s)^3 + t = 0$$
$$\sin(uv) + e^v + t^2 - 1 = 0.$$

Prove that near the origin $\mathbf{0} \in \mathbf{R}^4$, its solutions form the graph of a continuously differentiable function $G : \mathbf{R}^2 \to \mathbf{R}^2$. Clearly indicate the dependent and independent variables.

0.10. Let

$$f(x,y) = \begin{cases} \frac{yx^6+y^3+x^3y}{x^6+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) Show that all directional derivatives of f exist at $(0,0)$, and depend linearly on the vector we differentiate along.

b) Show that nevertheless, f is *not* differentiable at $(0,0)$.