

### August 2012 Tier 1 Analysis Exam

- Be sure to fully justify all answers.
- Scoring: Each one of the 10 problems is worth 10 points.
- Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write the problem number on each sheet of paper.
- Please be sure that you assemble your test with the problems presented in the correct order.

1. Let

$$f_n(x) = \sum_{k=1}^n (x^k - x^{2k}).$$

- (a) Show that  $f_n$  converges pointwise to a function  $f$  on  $[0, 1]$ .  
(b) Show that  $f_n$  does not converge uniformly to  $f$  on  $[0, 1]$ .

2. Define  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $f(x, y) = \frac{y^3 - \sin^3 x}{x^2 + y^2}$  if  $(x, y) \neq (0, 0)$  and  $f(0, 0) = 0$ .

- (a) Compute the directional derivative of  $f$  at  $(0, 0)$  for an arbitrary direction  $(u, v)$ .  
(b) Determine whether  $f$  is differentiable at  $(0, 0)$  and prove your answer.

3. Let  $E$  be a nonempty subset of a metric space and let  $f : E \rightarrow \mathbb{R}$  be uniformly continuous on  $E$ . Prove that  $f$  has a unique continuous extension to the closure of  $E$ . That is, there exists a unique continuous function  $g : \overline{E} \rightarrow \mathbb{R}$  such that  $g(x) = f(x)$  for  $x \in E$ .

4. Let  $B_r$  denote the ball  $B_r = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < r\}$  and let  $f : B_1 \rightarrow \mathbb{R}$  be a continuously differentiable function which is zero in the complement of a compact subset of  $B_1$ . Show that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{B_1 \setminus B_\varepsilon} \frac{x_1 f_{x_1} + x_2 f_{x_2}}{|\mathbf{x}|^2} dx_1 dx_2$$

exists and equals  $Cf(\mathbf{0})$  for a constant  $C$  which you are to determine.

5. Let  $E$  be a nonempty subset of a metric space and assume that for every  $\varepsilon > 0$   $E$  is contained in the union of finitely many balls of radius  $\varepsilon$ . Prove that every sequence in  $E$  has a subsequence which is Cauchy.

6. For which exponents  $r > 0$  is the limit

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n^2} \frac{n^{r-1}}{n^r + k^r}$$

finite? Prove your answer.

7. Let  $V$  be a neighborhood of the origin in  $\mathbb{R}^2$ , and  $f : V \rightarrow \mathbb{R}$  be continuously differentiable. Assume that  $f(0, 0) = 0$  and  $f(x, y) \geq -3x + 4y$  for  $(x, y) \in V$ . Prove that there is a neighborhood  $U$  of the origin in  $\mathbb{R}^2$  and a positive number  $\varepsilon$  such that, if  $(x_1, y_1), (x_2, y_2) \in U$  and  $f(x_1, y_1) = f(x_2, y_2) = 0$ , then

$$|y_2 - y_1| \geq \varepsilon |x_2 - x_1|.$$

8.

(a) Find necessary and sufficient conditions on functions  $h, k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that, given any smooth  $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of the form  $\mathbf{F} = (F_1(y, z), F_2(x, z), 0)$  and whose divergence is zero, there is a smooth  $\mathbf{G} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of the form  $\mathbf{G} = (G_1, G_2, 0)$  such that  $\nabla \times \mathbf{G} = \mathbf{F}$  in  $\mathbb{R}^3$  and  $\mathbf{G} = (h, k, 0)$  on  $z = 0$ . ( $\nabla \times \mathbf{G}$  is the curl of the vector field  $\mathbf{G}$ .)

(b) Let  $\mathbf{F}$  be as in (a) and evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dA$$

where  $S$  is the hemisphere

$$\{(x, y, z) : x^2 + y^2 + z^2 = 1, 0 \leq z \leq 1\},$$

$\mathbf{N}$  is the unit normal on  $S$  in the positive  $z$ -direction, and  $dA$  is the surface area element.

9. Let  $f = (f^1, \dots, f^n)$  map an open set  $U$  in  $\mathbb{R}^n$  into  $\mathbb{R}^n$  be  $C^1$  and suppose that, for some  $\bar{x} \in U$  the matrix  $f'(\bar{x})$  is negative definite (an  $n \times n$  matrix  $A$  is negative definite if  $\xi \cdot A\xi < 0$  for all nonzero  $\xi \in \mathbb{R}^n$ ). Show that there is a positive number  $\varepsilon$  and a neighborhood  $V$  of  $\bar{x}$  such that, if  $y_1, \dots, y_n$  are any  $n$  points in  $V$  and if  $A$  is the  $n \times n$  matrix whose  $i$ -th row is  $\nabla f^i(y_i)$ , then  $\xi \cdot A\xi \leq -\varepsilon |\xi|^2$  for all  $\xi \in \mathbb{R}^n$ .

10. Let  $f$  be a  $C^1$  mapping of an open set  $U \subset \mathbb{R}^n$  into  $\mathbb{R}^n$  and suppose that  $f(\bar{x}) = 0$  for some  $\bar{x} \in U$  and that  $f'(\bar{x})$  is negative definite. Show that there is a neighborhood  $W$  of  $\bar{x}$  and a positive number  $\delta$  such that, if a sequence  $\{x_k\}_{k=0}^{\infty}$  is generated from the recursion

$$x_{k+1} = x_k + \delta f(x_k)$$

with  $x_0 \in W$ , then each  $x_k$  is in  $W$  and  $x_k \rightarrow \bar{x}$  as  $k \rightarrow \infty$ . You may use here the result stated in problem 9 without having solved problem 9.