Tier 1 Analysis Exam: August 2011

Do all nine problems. They all count equally. Show all computations.

1. Let (X, d) be a compact metric space. Let $f : X \to X$ be continuous. Fix a point $x_0 \in X$, and assume that $d(f(x), x_0) \geq 1$ whenever $x \in X$ is such that $d(x, x_0) = 1$. Prove that $U \setminus f(U)$ is an open set in X, where $U = \{x \in X : d(x, x_0) < 1\}$.

2. Let $f_1 : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Define the sequence of functions $f_n : [a, b] \to \mathbb{R}$ by

$$f_{n+1}(x) = \int_a^x f_n(t)dt,$$

for each $n \geq 1$ and each $x \in [a, b]$. Prove that the sequence of functions

$$g_n(x) = \sum_{m=1}^{n} f_m(x)$$

converges uniformly on $[a, b]$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable everywhere. Assume $f(-\sqrt{2}, -\sqrt{2}) = 0$, and also that

$$\left|\frac{\partial f}{\partial x}(x, y)\right| \leq |\sin(x^2 + y^2)|$$

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \leq |\cos(x^2 + y^2)|$$

for each $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Prove that

$$|f(\sqrt{2}, \sqrt{2})| \leq 4.$$

4. Let $q_1, q_2, \ldots$ be an indexing of the rational numbers in the interval $(0, 1)$. Define the function $f(x) : (0, 1) \longrightarrow (0, 1)$, by

$$f(x) = \sum_{j : q_j < x} 2^{-j}.$$

(Here the sum is over all positive integers j such that $q_j < x$.)

a. Show that f is discontinuous at every rational number in $(0, 1)$.

b. Show that f is continuous at every irrational number in $(0, 1)$.

5. Show that the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\Phi(\theta, \phi) = (\sin\phi \cdot \cos\theta, \ \sin\phi \cdot \sin\theta),$$

is invertible in a neighborhood of $(\theta_0, \phi_0) = (\frac{\pi}{6}, \frac{\pi}{4})$ and find the partial derivatives of the inverse at the point $(\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4})$.

6. Let A be a domain in $\mathbf{R}^2$ whose boundary γ is a smooth, positively oriented curve.

a. Find a particular pair of functions $P : \mathbb{R}^2 \to \mathbb{R}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$ so that $\int_\gamma Pdx + Qdy$ equals the area of the domain A.

b. Let $|A|$ be the area of A. Find a function $R : \mathbb{R}^2 \to \mathbb{R}$ so that

$$\frac{1}{|A|} \int_\gamma Rdx + Rdy,$$

equals the average value of the square of the distance from the origin to a point of A.

7. Let C be a smooth simple closed curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C zdx - 2xdy + 3ydz$$

depends only on the orientation of C and on the area of the region enclosed by C but not on the shape of C or its location in the plane.

8. For each $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ define $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$. Consider

$$F(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^\lambda}, \quad \mathbf{x} \neq 0, \lambda > 0\,.$$

(i) Is there a value of λ for which F is divergence free?
(ii) Let $E : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$E(\mathbf{y}) = q\frac{\mathbf{y}}{|\mathbf{y}|^3}$$

where q is a positive real number. Let $S(\mathbf{x}, a)$ denote the sphere of radius $a > 0$ centered at $\mathbf{x}$. Assume $|\mathbf{x}| \neq a$. Compute

$$\int_{S(\mathbf{x},a)} E \cdot n \, dA$$

where dA is the surface area element and n is the unit outward normal on $S(\mathbf{x}, a)$.

9. Let $x_1 \in \mathbb{R}$. Define the sequence $(x_n)_{n \geq 2}$ by

$$x_{n+1} = x_n + \frac{\sqrt{|x_n|}}{n^2},$$

for each $n \geq 1$. Show that x_n is convergent.