TIER 1 ANALYSIS EXAM, JANUARY 2011

- Solve the following 10 problems, justifying all answers.
- Write the solution of each problem on a separate, clearly identified page.
- (1) In this problem we use the notation

$$|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

for the Euclidean norm of a vector $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$. Let $f:[0,1]\to\mathbb{R}^n$ be a continuous function. Show that

$$\left| \int_0^1 f(t) \, dt \right| \le \int_0^1 |f(t)| \, dt.$$

(2) A sequence

$$A_n = \left[\begin{array}{cc} a_n & b_n \\ c_n & d_n \end{array} \right], \quad n \geq 1,$$

of real matrices is said to converge if the sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$, $(d_n)_{n=1}^{\infty}$ converge. Fix a real matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and define

$$A_n = A - \frac{1}{3!}A^3 + \dots + \frac{(-1)^n}{(2n+1)!}A^{2n+1}, \quad n \ge 1.$$

Show that the sequence $(A_n)_{n=1}^{\infty}$ converges. (The limit is denoted $\sin(A)$.)

- (3) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function with the following property: for every positive integer n, and every $x, y \in \mathbb{R}$ such that $|x| + |y| > n^2$ and $|x y| < 1/n^2$, we have |f(x) f(y)| < 1/n. Show that f is uniformly continuous.
- (4) Determine the area enclosed by the curve

$$c(t) = (3\cos t - \cos(3t), 3\sin t - \sin(3t)), \quad t \in [0, 2\pi].$$

You can take it for granted that this is a simple curve.

- (5) Determine the volume of the solid $\{(x,y,z): \sqrt{x}+\sqrt{y}+\sqrt{z} \le 1, x,y,z \ge 0\}$.
- (6) Consider a differentiable, strictly decreasing function $f:[0,1] \to [0,1]$, and let $a \in [0,1]$ satisfy f(a) = a. (There obviously exists exactly one such point.) Assume that f'(a) < -1. Define a sequence $(x_n)_{n=0}^{\infty}$ by setting $x_0 = 0$ and $x_{n+1} = f(x_n)$ for $n \geq 0$. Show that the sequence $(x_n)_{n=0}^{\infty}$ does not converge.
- (7) Show that there exists a differentiable function f(x) defined in a neighborhood of $x_0 = \sqrt{2}$ such that $x^{f(x)} = f(x)$.
- (8) Given a sequence $f_n:[0,1]\to [0,1]$ of continuous functions, define $g_n:[0,1]\to\mathbb{R}$ by setting

$$g_n(x) = \int_0^1 \frac{f_n(t)}{(t-x)^{1/3}} dt, \quad x \in [0,1].$$

(Observe that this is an improper Riemann integral.) Show that the sequence $(g_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

(9) A sequence $(a_n)_{n=1}^{\infty}$ satisfies the inequality $|\sum_{k=1}^{n} a_k| \leq \sqrt{n}$ for all $n \geq 1$. Show that the series

$$\sum_{k=1}^{\infty} \frac{a_k}{k}$$

converges.

(10) Show that there does not exist a sequence $I_n = [a_n, b_n], n = 1, 2, \ldots$ of nonempty, pairwise disjoint intervals such that $\bigcup_n I_n = [0, 1]$.