Tier I Analysis Exam
August, 2010

- **Be sure to fully justify all answers.**
- **Scoring:** Each one of the 10 problems is worth 10 points.
- **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
- Please be sure that you assemble your test with the problems presented in correct order.

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(1) Let A and B be bounded sets of positive real numbers and let $AB = \{ab \mid a \in A, b \in B\}$. Prove that $\sup AB = (\sup A)(\sup B)$.

(2) A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *proper* if $f^{-1}(C)$ is compact for every compact set C. Prove or give a counterexample: if f and g are continuous and proper, then the product fg is proper.

(3) (a) Prove or give a counterexample: If $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function and $f(x) > x^2$ for all x, then given any $M \in \mathbb{R}$ there is an x_0 such that $|f'(x_0)| > M$.

(b) Prove or give a counterexample: If $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable function and $||f(x,y)|| > ||(x,y)||^2$ for all (x,y), then given any $M \in \mathbb{R}$ there is an $(x_0, y_0) \in \mathbb{R}^2$ such that $|\det(Df(x_0,y_0))| > M$.

(4) Suppose that $\{f_n\}$ is a sequence of continuous functions defined on the interval $[0,1]$ converging *uniformly* to a function f_0. Let $\{x_n\}$ be a sequence of points converging to a point x_0 with the property that for each n, $f_n(x_n) \geq f_n(x)$ for all $x \in [0,1]$. Prove that $f_0(x_0) \geq f_0(x)$ for all $x \in [0,1]$.

(5) Let f be continuous at $x = 0$, and assume

$$\lim_{x\to 0} \frac{f(2x) - f(x)}{x} = L.$$

Prove that $f'(0)$ exists and $f'(0) = L$.

(6) Let $R = \{(x,y) \mid 0 \leq x,\ 5|y| \leq 3|x|,\ x^2 - y^2 \leq 1\}$, a compact region in $\mathbb{R}^2$. For some region $S \subset \mathbb{R}^2$, the function $F\colon S \to R$ given by $F(r,\theta) = (r\cosh\theta, r\sinh\theta)$ is one-to-one and onto. Determine S and use this change of variable to compute the integral

$$\iint_R \frac{dx\,dy}{1 + x^2 - y^2}.$$

(Recall that $\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$ and $\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$.)

(7) Let $d(x) = \min_{n\in\mathbb{Z}} |x - n|$, where $\mathbb{Z}$ is the set of all integers.

(a) Prove that $f(x) = \sum_{n=0}^{\infty} \frac{d(10^n x)}{10^n}$ is a continuous function on $\mathbb{R}$.

(b) Compute explicitly the value of $\int_0^1 f(x)dx$.

(8) Suppose f and φ are continuous real valued functions on $\mathbb{R}$. Suppose $\varphi(x) = 0$ whenever $|x| > 5$, and suppose that $\int_{\mathbb{R}} \varphi(x)dx = 1$. Show that

$$\lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}} f(x-y)\varphi\left(\frac{y}{h}\right) dy = f(x)$$

for all $x \in \mathbb{R}$.

(9) Let $f(x, y, z)$ and $g(x, y, z)$ be continuously differentiable functions defined on $\mathbb{R}^3$. Suppose that $f(0, 0, 0) = g(0, 0, 0) = 0$. Also, assume that the gradients $\nabla f(0, 0, 0)$ and $\nabla g(0, 0, 0)$ are linearly independent. Show that for some $\epsilon > 0$ there is a differentiable curve $\gamma\colon (-\epsilon, \epsilon) \to \mathbb{R}^3$ with nonvanishing derivative such that $\gamma(0) = (0, 0, 0)$ and $f(\gamma(t)) = g(\gamma(t)) = 0$ for all $t \in (-\epsilon, \epsilon)$.

(10) Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \text{ and } z = e^{x^2+2y^2}\}$. So, S is that part of the surface described by $z = e^{x^2+2y^2}$ that lies inside the cylinder $x^2 + y^2 = 1$. Let the path $C = \partial S$. Choose (specify) an orientation for C and compute

$$\int_C (-y^3 + xz)dx + (yz + x^3)dy + z^2 dz\ .$$