TIER 1 ANALYSIS EXAM
AUGUST 2007

(1) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$f(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that is differentiable at all points $(x, y) \in \mathbb{R}^2$ except $(0, 0)$. Show that f is not differentiable at $(0, 0)$.

(2) Given $\lambda \in \mathbb{R}$, define $h_\lambda : \mathbb{R}^2 \to \mathbb{R}$ by

$$h_\lambda(x, y) \;=\; -x^4 \;+\; x^2 \;+\; y^2 \;+\; \lambda \cdot \sin(x \cdot y).$$

For which values of λ does h_λ have a local minimum at $(0, 0)$? Justify your answer.

(3) Let $\gamma \subset \mathbb{R}^2$ be the simple closed curve described in polar coordinates by $r = \cos(2\theta)$ where $\theta \in [-\pi/4, \pi/4]$. Suppose that γ is positively oriented. Compute the line integral

$$\int_\gamma 3y\; dx \;+\; x\; dy.$$

Provide the details of your computation.

(4) Let X be a metric space such that $d(x, y) \leq 1$ for every $x, y \in X$, and let $f : X \to \mathbb{R}$ be a uniformly continuous function. Does it follow that f must be bounded? Justify your answer with either a proof or a counterexample.

(5) Let

$$f(x, y) \;=\; (x + e^{2y} - 1,\; \sin(x^2 + y)),$$

and let

$$h(x, y) \;=\; (1 + x)^5 \;-\; e^{4y}.$$

Show that there exists a continuously differentiable function $g(x, y)$ defined in a neighborhood of $(0, 0)$ such that $g(0, 0) = 0$ and $g \circ f = h$. Compute $\frac{\partial g}{\partial y}(0, 0)$.

(6) Let $c_1, c_2, \ldots$ be an infinite sequence of distinct points in the interval $[0,1]$. Define $f : [0,1] \to \mathbb{R}$ by setting $f(x) = 1/n$ if $x = c_n$ and $f(x) = 0$ if $x \notin \{c_n\}$. State the definition of a Riemann integrable function, and directly use this definition to show that

$$\int_0^1 f(x)\, dx$$

exists.

(7) Show that the formula

$$g(x) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^2}\, e^{\int_0^x t \sin(\frac{n}{t})\, dt}$$

defines a function $g : \mathbb{R} \to \mathbb{R}$. Prove that g is continuously differentiable.

(8) Consider an unbounded sequence $0 < a_1 < a_2 < \cdots$, and set

$$s \;=\; \limsup_{n\to\infty} \frac{\log n}{\log a_n}.$$

Show that the series

$$\sum_{n=1}^{\infty} a_n^{-t}$$

converges for $t > s$ and diverges for $t < s$.

(9) Define a sequence $\{a_n\}$ by setting $a_1 = 1/2$ and $a_{n+1} = \sqrt{1 - a_n}$ for $n \geq 2$. Does the sequence a_n converge? If so, what is the limit? Justify your answer with a proof.