Tier I Analysis Exam
January 2007

Notations: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the standard scalar (inner) product $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$ and the Euclidean norm $|x| = \sqrt{\langle x, x\rangle}$.

1. Use $\varepsilon - \delta$ notation to state the condition that a function $f : \mathbb{R} \to \mathbb{R}$ is not continuous on $\mathbb{R}$.

2. For $A > 0$ consider the sequence $x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right)$ $(n = 1, 2, \ldots)$ with $x_1 > 0$. Show that $\{x_n\}$ converges and find its limit.

3. Let $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ and
$X = \{f \in C(\Omega) : |f(x) - f(y)| \le |x - y| \quad \forall\, x, y \in \Omega, |f(0)| < 1\}$
where $C(\Omega)$ denotes the space of continuous functions f from Ω to $\mathbb{R}$ (both with standard, Euclidean topologies), with sup-norm $\|f\| := \sup_{X\in\Omega} |f(x)|$. Show that X is a sequentially compact subset of $C(\Omega)$, i.e., every sequence $\{f_n\}$, $f_n \in X$, has a convergent subsequence f_{n_j} converging to f_∞ in X in the sup-norm topology.

4. Let $\sum_{n=0}^{\infty} a_n$ be a convergent series with nonnegative terms, and S be its sum. For $f(x) = \sum_{n=0}^{\infty} a_n x^n$ show that $\lim_{x\to+1^-} f(x) = S$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and $\lim_{x\to-\infty} f'(x) = -\infty$, $\lim_{x\to+\infty} f'(x) = +\infty$. Show that for any $A \in \mathbb{R}$ there exists $a \in \mathbb{R}$ such that $f'(a) = A$.
Warning: $f'(x)$ may not be continuous.

6. Consider the function

$$K(t,x) = xt^{-\frac{3}{2}}e^{-\frac{x^2}{4t}}$$

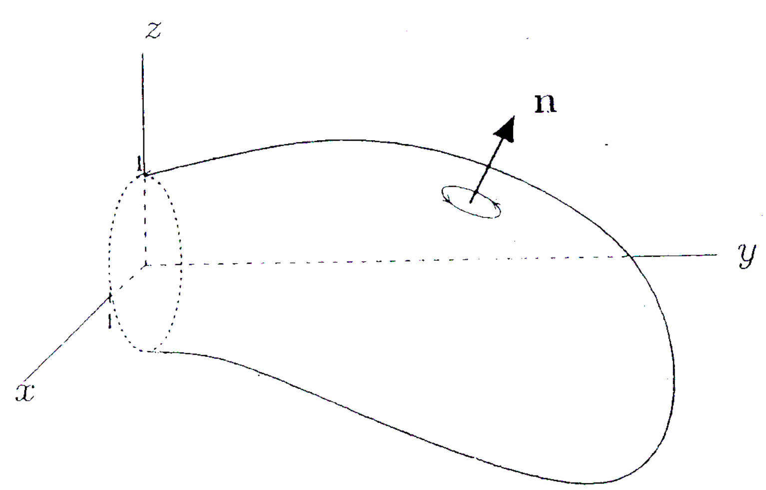
defined for all $x \in \mathbb{R}$ and $t > 0$. Clearly, $K(t,0) = 0$ for $t > 0$. Show that $K(t,x) \to 0$ as $t \to 0^+$ for any fixed x. Can you define K at $(0,0)$ to make it continuous there?

7. Calculate

$$\int\int_D \cos\left(\frac{x+2y}{-x+y}\right) dxdy,$$

where D is the triangular region in $\mathbb{R}^2$ having vertices $(0,0), (-2,4), (-3,3)$.

8. A soap film bubble blown from a circular hoop describes an undetermined region $\Omega \subset \{(x,y,z) \in \mathbb{R}^3 : y \leq 0\}$ having three-dimensional volume equal to 10. Let S denote that portion of $\partial\Omega$ comprised of the soap film (it does not include the unit disk D in the x,z–plane). Suppose a force field $\mathbf{F} = (z^2, 3y+5, x^3)$ is applied. Find $\int_S \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{n}$ is the outward pointing normal on $\partial\Omega$, and dA is the surface element.



9. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $f(x) = 0$ if $|x| \geq 1$.

a) Show that the improper integral

$$g(y) = \int_{-\infty}^{\infty} \frac{f(x)}{\sqrt{|x-y|}} dx$$

converges for all $y \in \mathbb{R}$, and $g(y)$ is continuous.

b) Show that, if additionally f is continuously differentiable, then so is g and

$$g'(y) = \int_{-\infty}^{\infty} \frac{f'(x)}{\sqrt{|x-y|}} dx$$

10. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map and $df_a : \mathbb{R}^n \to \mathbb{R}^n$ be its differential at $a \in \mathbb{R}^n$. Suppose that df_a is positive at any $a \in \mathbb{R}^n$, in the sense that $\langle df_a(x), x\rangle > 0$ for all $a \in R^n$, and $x \in \mathbb{R}^n - \{0\}$.

Prove that f is injective

Hint: For $a \in \mathbb{R}^n - \{0\}$ consider $g : \mathbb{R} \to \mathbb{R}^n$ defined by $g(t) = f(ta)$. Find $g'(t)$ and show that $f(a) \neq f(0)$.