

Tier 1 Analysis Exam
August 2006

1. Let $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function.
 - (1) Prove that if f is uniformly continuous, and if $\{p_k\}$ is a Cauchy sequence in A , then $\{f(p_k)\}$ is a Cauchy sequence in \mathbb{R}^m .
 - (2) Give an example of continuous f and a Cauchy sequence $\{p_k\}$ in some A (you may take $n = m = 1$) for which $\{f(p_k)\}$ is not a Cauchy sequence.
2. Let $f : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Assume f is differentiable everywhere in (a, b) , except possibly at a point c . Show that, if $\lim_{x \rightarrow c} f'(x)$ exists and is equal to L , then f is differentiable at c and $f'(c) = L$.
3. Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 function with $g(\frac{1}{3}, \frac{2}{3}) = 3$, $\frac{\partial g}{\partial r}(\frac{1}{3}, \frac{2}{3}) = -1$, and $\frac{\partial g}{\partial s}(\frac{1}{3}, \frac{2}{3}) = -4$, where (r, s) are the coordinates for the \mathbb{R}^2 . Define $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$f(x, y, z) = g\left(\frac{x}{z}, \frac{y}{z}\right),$$

for $z \neq 0$. Show that the level surface $f^{-1}(3)$ has a tangent plane at the point $(1, 2, 3)$ and find a linear equation for it.

4. For which positive integers k does the series

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/k)}{n}$$

converge? Justify your answer with a proof.

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Let $x \in \mathbb{R}$ and define the sequence $\{x_n\}$ inductively by setting $x_0 = x$ and $x_{n+1} = f(x_n)$. Suppose that $\{x_n\}$ is bounded. Prove that there exists $y \in \mathbb{R}$ such that $f(y) = y$.
6. Decide whether or not the function $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{x}{q}, & \text{if } x \notin \mathbb{Q} \text{ and } y = \frac{p}{q} \in \mathbb{Q} \\ \frac{y}{q}, & \text{if } y \notin \mathbb{Q} \text{ and } x = \frac{p}{q} \in \mathbb{Q} \\ 0, & \text{if } (x, y) \in \mathbb{Q} \times \mathbb{Q} \text{ or } (x, y) \in \mathbb{R} \setminus \mathbb{Q} \times \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is Riemann integrable on $[0, 1] \times [0, 1]$. Prove your decision from the definition without invoking any theorems about integrable functions. (Here all fractions $\frac{p}{q}$ are assumed to be reduced.)

7. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 function (i.e. f has continuous second order partial derivatives). Suppose $p_0 \in \mathbb{R}^n$ is a critical point of f . If

$$\det \left[\frac{\partial^2 f}{\partial x_i \partial x_j} (p_0) \right] \neq 0,$$

show that p_0 is isolated, i.e. there is a neighborhood of p_0 in which p_0 is the only critical point of f .

8. Prove that $\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ converges pointwise but not uniformly on \mathbb{R} . Let $f(x) = \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$.

Is it true that the Riemann integral $\int_0^1 f(x) dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x}{1+n^2x^2} dx$? Justify.

9. Let $\{a_n\}$ be a sequence. Show that

$$\limsup_{n \rightarrow \infty} \frac{a_n}{n} \leq \limsup_{n \rightarrow \infty} (a_n - a_{n-1}).$$

10. Let $Q \subset \mathbb{R}^3$ be any solid rectangular box with one vertex at the origin. Show that

$$\int_{\partial Q} \frac{\mathbf{x} \cdot \hat{\mathbf{n}}}{\|\mathbf{x}\|^3} dS = \frac{\pi}{2}.$$

Here $\hat{\mathbf{n}}$ is the unit outer normal on ∂Q and dS is the area element. (You should notice that this integral is not an improper integral.)