Tier I Analysis Exam
August, 2005

Justify your answers. All problems carry equal weight.

1. Let $p > 0$. Evaluate
$$\lim_{n\to\infty} \frac{1^p + 3^p + \ldots + (2n-1)^p}{n^{p+1}}.$$

2. For $x > 0$, define
$$\phi(x) = \begin{cases} 0 & \text{if} \quad x \text{ is irrational} \\ \frac{1}{q} & \text{if} \quad x = \frac{p}{q} \end{cases}$$
where p, q have no common factor and $q \geq 1$.

   (a) Where is ϕ continuous?

   (b) Where is ϕ differentiable?

3. Let a, b be real with $|b| > \max\{1, |a|\}$. For $x \in \mathbf{R}$, define
$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(a^n x)}{b^n}.$$

   (a) Show that f is uniformly continuous on all of $\mathbf{R}$.

   (b) Let
$$\gamma = \{(x, y) : y = f(x), \ \ 0 \leq x \leq 1\}$$
   be the graph of f over the unit interval. Show that γ has finite length.

4. Let M_2 denote the set of 2-by-2 matrices with real entries, and for $A \in M_2$, define $S(A) = A^2$. Does the mapping $S : M_2 \to M_2$ have a local inverse near the identity matrix?

5. Fix $a > 0$. Let $x_1, \ldots, x_n$ be non-negative numbers with
$$\sum_{i=1}^{n} x_i = na.$$
   Show that
$$\sum_{i<j} x_i x_j \leq \frac{1}{2} n(n-1)a^2.$$

Please turn over.

6. Let p be real. Suppose $f : \mathbf{R}^n - \{0\} \to \mathbf{R}$ is continuously differentiable, and satisfies

$$f(\lambda x) = \lambda^p f(x), \quad \text{for all } \; x \neq 0 \; \text{ and for all } \; \lambda > 0.$$

Let $\nabla f(x)$ denote the gradient of f at x and $\cdot$ the Euclidean inner product. Prove that

$$x \cdot \nabla f(x) = p f(x), \quad x \neq 0.$$

7. A family $\mathcal{F}$ of continuous real-valued functions of a real variable is called *equicontinuous at* x if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $f \in \mathcal{F}$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

$\mathcal{F}$ is called *equicontinuous on the set* E if it is equicontinuous at each point x of E. (Note: the constant δ may depend on both ϵ and x.)

Now suppose $\mathcal{F}$ is a family of continuous real-valued functions defined on an open interval $I \subseteq \mathbf{R}$, and let $x_0 \in I$.

(a) Suppose $\mathcal{F}$ is equicontinuous at every point of $I \setminus \{x_0\}$. Must $\mathcal{F}$ also be equicontinuous at x_0?

(b) Suppose $\mathcal{F}$ is equicontinuous at x_0. Must $\mathcal{F}$ also be equicontinuous at every point in some neighborhood J of x_0?

8. Let U be an open subset of $\mathbf{R}^n$ and $f : U \to \mathbf{R}^n$ be differentiable. Suppose there exists $C > 0$ such that

$$|f(x) - f(y)| \geq C|x - y|$$

for all $x, y \in U$. Let $df(x)$ denote the Jacobian derivative of f at x (that is, the linear mapping given by the n by n matrix of partial derivatives). Show that $\det df(x) \neq 0$ for all $x \in U$.

9. Let $m > 0$ be a real number, let $r = (x^2 + y^2 + z^2)^{1/2}$, and consider the vector field on $\mathbf{R}^3$ given by $\vec{F} = r^m \cdot (x, y, z) = (x^2 + y^2 + z^2)^{m/2}(x, y, z)$.

(a) Compute the divergence $\operatorname{div}(\vec{F})$.

(b) Using part (a) and the Divergence Theorem, calculate

$$\iiint_{B^3} r^m \, dV$$

where $B^3 = \{(x, y, z) : r \leq 1\}$ is the closed unit ball centered at the origin and $dV = dx\, dy\, dz$ is the Euclidean volume.

End of exam.