Tier I exam in analysis - August 2003

Answer all the problems. Justify your answers.

1. Let $f(x)$ be a function that is continuous in $[-1,1]$, differentiable in $(-1,1)$, and satisfies $f(-1) = -\pi/2$, $f(1) = \pi/2$, $f'(x) \geq \frac{1}{\sqrt{1-x^2}}$ in $(-1,1)$. Prove that $f(x) = \arcsin(x)$ in $[-1,1]$.

2. Determine the values of $x \in \mathbf{R}$ such that the series
$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n+x^2)}}$$
converges.

3. Recall that a square matrix M is called orthogonal if its rows form an orthonormal set. The set of all orthogonal matrices will be denoted by O. (Note that an orthogonal matrix necessarily satisfies the condition $MM^t = I$ where M^t denotes the transpose of M.)
$$\text{Let } M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
be a given element of O where a, b, c and d are real numbers.

   (i). Prove that, except for 4 special matrices M_0, there always exists a number $\delta > 0$ and three functions f, g, h, continuously differentiable for $x \in (a-\delta, a+\delta)$, such that
$$\begin{pmatrix} x & f(x) \\ g(x) & h(x) \end{pmatrix} \in O,$$
for all $x \in (a-\delta, a+\delta)$ with $f(a) = b$, $g(a) = c$ and $h(a) = d$.

   (ii). What are the four exceptional matrices of part (i)?

4. A vector field $\vec{F} : \mathbf{R^2} \to \mathbf{R^2}$ is said to be conservative in an open set D if the line integral $\int_C \vec{F} \cdot ds = 0$ for every closed curve $C \subset D$. Find all numbers a and b such that the vector field
$$\vec{F}(x,y) = \left(\frac{x+ay}{x^2+y^2}, \frac{bx+y}{x^2+y^2}\right)$$
is conservative in
$$D = \{(x,y) : \frac{1}{9} < x^2+y^2 < \frac{1}{4}\}.$$

5. Consider the triangle with vertices $(3,0)$ $(5,0)$ and $(5,1)$ in the (x,y)-plane. Revolve it around the y-axis in (x,y,z)-space $\mathbf{R^3}$ to sweep out a "triangular torus" T, evaluate the surface integral
$$\int_T \vec{v} \cdot \vec{n}\, dS .$$
Here $\vec{v} : \mathbf{R^3} \to \mathbf{R^3}$ is the vector field $\vec{v}(x,y,z) = (-y, x, z)$, $\vec{n}$ is the outward unit normal field on T, and dS is the usual surface-area element on T.

6. Suppose $f\colon \mathbf{R}^2 \to \mathbf{R}$ is a C^∞ function that has a critical point at $(0,0)$. Suppose that $f(0,0) = 0$ and that all the first and second order partial derivatives of f vanish at $(0,0)$. Also, assume that not all third order partial derivatives vanish at $(0,0)$. Show that f can have neither a local max nor a local min at the critical point $(0,0)$.

7. Recall that a function $g\colon [a,b] \to \mathbf{R}$ is said to be Lipschitz if there is a constant K such that $|g(x) - g(y)| \leq K\,|x - y|$ for all $x, y \in [a,b]$.

   Assume that f is a bounded Riemann integrable function on $[a,b]$. Prove that for each $\varepsilon > 0$, there exists a Lipschitz function g such that

$$\int_a^b |f(x) - g(x)|\, dx < \varepsilon.$$

8. (a) If $B \subset \mathbf{R}^n$ is a bounded set and $f\colon B \to \mathbf{R}$ is uniformly continuous, show that $f(B)$ is bounded.

   (b) Give an example to show that the conclusion of part (a) is not necessarily true if f is merely continuous on B.