Name ______________________________ **ID number** ________________________

Analysis Qualifying Exam, Spring 2002, Indiana University

Instructions. There are nine problems, each of equal value. Show your work, justifying all steps by direct calculation or by reference to an appropriate theorem. Good luck!

1. Let $a_0, a_1, ..., a_n$ be a set of real numbers satisfying

$$a_0 + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0.$$

Prove that the polynomial $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ has at least one root in $(0, 1)$.

2. Let $f_n : R \to R$ be differentiable, for all n, with derivative uniformly bounded (in absolute value) by 1. Further assume that $\lim_{n\to\infty} f_n(x) = g(x)$ exists for all $x \in R$. Prove that $g : R \to R$ is continuous.

3. Let $f : R^2 \to R$ have the property that for every $(x, y) \in R^2$, there exists *some* rectangular interval $[a, b] \times [c, d]$, $a < x < b$, $c < y < d$, on which f is Riemann integrable. Show that f is Riemann integrable on *any* rectangular interval $[e, f] \times [g, h]$.

4. Show that the sequence

$$1/2, (1/2)^{1/2}, ((1/2)^{1/2})^{1/2}, (((1/2)^{1/2})^{1/2})^{1/2}, \ldots$$

converges to a limit L, and determine this limit.

5. Let f, $g : R^2 \to R$ be functions with continuous first derivative such that the map $F : (x, y) \to (f, g)$ has Jacobian determinant

$$\det \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

identically equal to one. Show that F is open, i.e., it takes open sets to open sets. If also f is *linear* , i.e. f_x and f_y are constant, show that F is one-to-one.

6. Let $f : (0, 1] \to R$ have continuous first derivative, with $f(1) = 1$ and $|f'(x)| \le x^{-1/2}$ if $|f(x)| \le 3$. Prove that $\lim_{x\to 0^+} f(x)$ exists.

7. Letting $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in R^3, evaluate the surface integral

$$F = -\int\int_S P(x, y, z)\nu \, dA,$$

where $\nu(x, y, z) = (x, y, z)$ denotes the outward normal to S, dA the standard surface element, and:

(a) $P(x, y, z) = P_0$, P_0 a constant.

(b) $P(x, y, z) = Gz$, G a constant.

Remark (not needed for solution): F corresponds to the total buoyant force exerted on the unit ball by an external, ideal fluid with pressure field P.

8. Compute the integral

$$\int_C y(z+1)dx + xzdy + xydz,$$

where $C: \; x = \cos\theta, \, y = \sin\theta, \, z = \sin^3\theta + \cos^3\theta, \quad 0 \leq \theta \leq 2\pi.$

9. Let X and Y be metric spaces and $f : X \to Y$. If $\lim_{p\to x} f(p)$ exists for all $x \in X$, show that $g(x) = \lim_{p\to x} f(p)$ is continuous on X.

Tier 1 Analysis Examination – August, 2002

1. In the classical *false position* method to find roots of $f(x) = 0$, one begins with two approximations $x_0,\ x_1$ and generates a sequence of (hopefully) better approximations via

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_0}{f(x_n) - f(x_0)} \quad \text{for} \quad n = 1, 2, \ldots$$

Consider the following sketch in which the function $f(x)$ is to be increasing and convex:

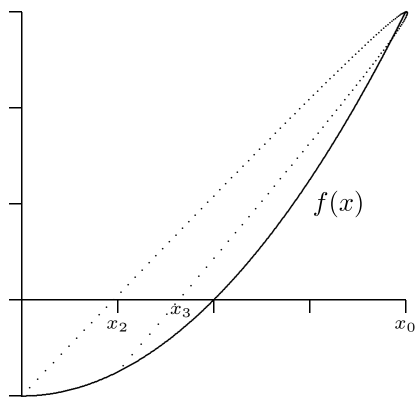

Fig. 1.2

The sequence $\{x_n\}$ is constructed as follows. We begin with the two approximations $(x_0, f(x_0))$ and $(x_1, f(x_1)) = (0, f(0))$ The chord is drawn between these two points; the point at which this chord crosses the x–axis is taken to be the next approximation x_2. One then draws the chord between the two points $(x_0, f(x_0))$ and $(x_2, f(x_2))$. The next approximation x_3 is that point where this chord crosses the axis, as shown. For f strictly increasing and convex and for initial approximations $x_0 > 0$, $x_1 = 0$ with $f(x_0) > 0$, $f(x_1) < 0$, prove *rigorously* that this sequence must converge to the unique solution of $f(x) = 0$ over $[x_1, x_0]$.

2. (a) Show that it is possible to solve the equations

$$xu^2 + yzv + x^2z - 3 = 0$$
$$xyv^3 + 2zu - u^2v^2 - 2 = 0$$

for (u, v) in terms of (x, y, z) in a neighborhood of $(1, 1, 1, 1, 1)$.

(b) Given that the inverse of the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

find $\frac{\partial u}{\partial x}$ at $(1, 1, 1)$.

3. Let X be a complete metric space and let Y be a subspace of X. Prove that Y is complete if and only if it is closed.

4. Suppose $f\colon K \to \mathbb{R}^1$ is a continuous function defined on a compact set K with the property that $f(x) > 0$ for all $x \in K$. Show that there exists a number $c > 0$ such that $f(x) \geq c$ for all $x \in K$.

5. Let $f(x)$ be a continuous function on $[0, 1]$ which satisfies

$$\int_0^1 x^n f(x)\, dx = 0 \quad \text{for all} \quad n = 0, 1, \ldots$$

Prove that $f(x) = 0$ for all $x \in [0, 1]$.

6. Show that the Riemann integral $\int_0^\infty \frac{\sin x}{x}\, dx$ exists.

7. Let

$$G(x, y) = \begin{cases} x(1-y) & \text{if } 0 \le x \le y \le 1 \\ y(1-x) & \text{if } 0 \le y \le x \le 1 \end{cases}$$

Let $\{f_n(x)\}$ be a uniformly bounded sequence of continuous functions on $[0, 1]$ and consider the sequence

$$u_n(x) = \int_0^1 G(x, y) f_n(y)\, dy.$$

Show that the sequence $\{u_n(x)\}$ contains a uniformly convergent subsequence on $[0, 1]$.

8. Let f be a real–valued function defined on an open set $U \subset \mathbb{R}^2$ whose partial derivatives exist everywhere on U and are bounded. Show that f is continuous on U.

9. For $x \in \mathbb{R}^3$ consider spherical coordinates $x = r\omega$ where $|\omega| = 1$ and $|x| = r$. Let ω_k be the k'th component of ω for any $k = 1, 2, 3$. Use the divergence theorem to evaluate the surface integral

$$\int_{|\omega|=1} \omega_k\, dS.$$

10. Let $\{f_k\}$ be a sequence of continuous functions defined on $[a, b]$. Show that if $\{f_k\}$ converges uniformly on (a, b), then it also converges uniformly on $[a, b]$.

11. Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be a continuous mapping. Show that $f(S)$ is bounded in $\mathbb{R}^k$ if S is a bounded set in $\mathbb{R}^n$.