Tier I Analysis Exam, Fall 2000

It is important to justify your answers. A correct answer, without justification (for example to #3 or #4) will receive no credit.

1. Evaluate the limit

$$\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+n}\right\}$$

by interpreting it as a definite integral.

2. Consider the 1-form F defined on $\mathbb{R}^2\backslash\{0\}$ by

$$F=\frac{xdy-ydx}{x^2+y^2}$$

(a) Evaluate $\int_{\partial C} F$, where C is the unit square, $[-1,1]\times[-1,1]$, in $\mathbb{R}^2$, positively oriented.

(b) Is F exact on $\mathbb{R}^2\backslash\{0\}$?

3. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous, real-valued functions on $[0,1]$ that converges uniformly to a function f on $[0,1]$. Must f have a zero in $[0,1]$ (i.e. $f(x)=0$ for some $x\in[0,1]$) if each f_n has a zero in $[0,1]$?

4. Does the series $\sum_{n=1}^{\infty}\frac{\cos(\log n)}{n}$ converge or diverge?

5. Let f be a continuous function on $[0,1]$. Show that

$$\int_0^1 f(x)\ \sin(nx)\ dx\to 0$$

as $n\to\infty$.

6. Is the function $f(x)=\sqrt{x}$ uniformly continuous on $[0,\infty)$?

7. Consider the function $f=\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Show that $\frac{\partial f}{\partial x}$ exists everywhere on $\mathbb{R}^2$ but that $\frac{\partial f}{\partial x}$ is not continuous everywhere.

8. Let

$$X = \left\{ \begin{array}{l} f : [0, 2\pi] \to \mathbb{R} : f \text{ is continuous} \\ \text{and } |f(x)| \leq 1 \text{ for all } x \in [0, 2\pi] \end{array} \right\}$$

Put a metric d on X by defining

$$d(f, g) = \sqrt{\int_0^{2\pi} (f(x) - g(x))^2 \, dx}$$

(You may assume that d actually does define a metric on X.)

Is (X, d) compact?

9. Let $\mathbb{R}^{2\times 2}$ denote the set of all real 2×2 matrices. Make it a metric space by identifying $\mathbb{R}^{2\times 2}$ with the 4-dimensional Euclidean space $\mathbb{R}^4$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \approx (a, b, c, d)$$

Let $X \subset \mathbb{R}^{2\times 2}$ denote the subset of all invertible 2×2 matrices. Is X connected?

10. Consider the two equations

$$F_1(x, y, u, v) \equiv e^{x^2 - y^2} u^5 - v^3 = 0$$

$$F_2(x, y, u, v) \equiv e^{u^2 - v^2} x^2 - y^2 = 0$$

Prove that there exists a neighborhood, U, of $(1, 1) \in \mathbb{R}^2$ and functions $u(x, y)$ and $v(x, y)$ on U with $u(1, 1) = v(1, 1) = 1$ such that

$$F_1\left(x, y, u(x, y), v(x, y)\right) = F_2\left(x, y, u(x, y), v(x, y)\right) = 0 \qquad \forall x, y \in U.$$

Find $\left.\frac{\partial u(x,y)}{\partial x}\right|_{(1,1)}$.

11. Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n = 1, 2, \ldots)$ be an equicontinuous sequence of functions. If $f_n(x) \to 0$ as $n \to \infty$ for each $x \in \mathbb{R}$, does it follow that the convergence is uniform on $\mathbb{R}$?