Tier I Analysis Examination

August 1999

There is no unusual notation in this exam: $\mathbf{R}$ stands for the real line, $\mathbf{R}^n$ for n-dimensional Euclidean space, and $\|x\|$ for the Euclidean norm of a vector $x \in \mathbf{R}^n$ (distance from x to 0). You must do eight of the following problems. Please indicate which of the nine problems should not be graded.

1. Let $f : (0, \infty) \to \mathbf{R}$ be a bounded continuous function. Show that there exists $c \in (0, \infty)$ such that $\int_0^\infty e^{-x} f(x)\, dx = f(c)$.
2. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a a continuous function such that $\|f(x)\| < \|x\|$ for every point $x \neq 0$. Fix a point $x_1 \in \mathbf{R}^n$, and define recursively $x_{n+1} = f(x_n)$ for $n \geq 1$. Show that the sequence $(x_n)_{n=1}^\infty$ converges to 0.
3. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a mapping of class C^1 such that the Jacobian determinant $J_f(x)$ is different from zero for all points x. Assume in addition that $\{x : \|f(x)\| < M\}$ is a bounded set for every $M > 0$. Show that f is onto. That is, show that for every $y \in \mathbf{R}^n$ there exists at least one point $x \in \mathbf{R}^n$ such that $f(x) = y$.
4. Consider the surface S surface in $\mathbf{R}^3$ consisting of all points of coordinates (x, y, z) such that $x^2 + y^2 + z^2 = 1$ and $x \geq \frac{1}{2}$, and choose an orientation for S. Calculate the integral $\int_S \omega$, where the 2-form ω is defined by

   $$\omega(x, y, z) = x dx \wedge dy + y dy \wedge dz + z dz \wedge dx$$

   for $(x, y, z) \in \mathbf{R}^3$.
5. Denote by $D = \{(x, y) : x > 0\}$ the right half-plane in $\mathbf{R}^2$, and let f be a function of class C^1 defined on D. Assume that

   $$\frac{\partial f}{\partial x}(x, y) \leq \frac{1}{\sqrt{x}} \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) \leq 1$$

   for all $(x, y) \in D$. Show that f is uniformly continuous on D.
6. Assume that the function $f : \mathbf{R} \to \mathbf{R}$ is differentiable at every point, and $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are two sequences converging to zero with $a_n < b_n$ for all n. Do the quotients

   $$\frac{f(b_n) - f(a_n)}{b_n - a_n}$$

   necessarily converge to $f'(0)$? (Prove if yes, give a counterexample if no.)
7. Let $f : [0, 1] \times [0, 1] \to \mathbf{R}$ be a continuous function, and define $g : [0, 1] \to \mathbf{R}$ by $g(x) = \max_{y \in [0,1]} f(x, y)$. Show that g is continuous.
8. Let $(a_n)_{n=1}^\infty$ be a sequence of real numbers, and define $s_n = \sum_{k=1}^n a_k$. Assume that $\lim_{n \to \infty} \sqrt{n} a_n = 1$ and prove that $\lim_{n \to \infty} s_n / \sqrt{n} = 2$.
9. Consider a complete metric space (X, d), and a sequence $F_1 \supseteq F_2 \supseteq \cdots$ of nonempty, closed subsets of X. Assume that for each n, the set F_n can be covered by a finite number of balls of radius $1/n$. For each n, select a point $x_n \in F_n$. Prove that the sequence $(x_n)_{n=1}^\infty$ has a convergent subsequence.