Tier 1 Analysis Examination

January 1999

1. Prove that the function

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

satisfies $f'(0) > 0$, but that there is no open interval containing 0 on which f is increasing.

2. Let $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a mapping defined by $F(x, y) = (u, v)$ where

$$u = u(x, y) = x\cos(y)$$
$$v = v(x, y) = y\cos(x).$$

Note that $F(-\pi/3, \pi/3) = (-\pi/6, \pi/6)$.

(i) Show that there exist neighborhoods U of $(-\pi/3, \pi/3)$, V of $(-\pi/6, \pi/6)$, and a differentiable function $G\colon V \to U$ such that F restricted to U is one-to-one, $F(U) = V$ and $G(F(x, y)) = (x, y)$ for every $(x, y) \in U$.

(ii) Let U, V and G be as in part (i), and write

$$G(u, v) = (x, y), \text{ with } x = x(u, v), y = y(u, v).$$

Find

$$\frac{\partial x}{\partial u}(-\pi/6, \pi/6) \quad \text{and} \quad \frac{\partial y}{\partial v}(-\pi/6, \pi/6).$$

3. Beginning with $a_1 \geq 2$, define a sequence recursively by $a_{n+1} = \sqrt{2 + a_n}$. Show that the sequence is monotone and compute its limit.

4. Let $f\colon K \to \mathbf{R}^n$ be a one-to-one continuous mapping, where $K \subset \mathbf{R}^n$ is a compact set. Thus, the mapping f^{-1} is defined on $f(K)$. Prove that f^{-1} is continuous.

5. Let S denote the 2-dimensional surface in $\mathbf{R}^3$ defined by $F\colon D \to \mathbf{R}^3$ where $D = \{(x, y) : x^2 + y^2 \leq 4\}$ and $F(x, y) = (x, y, 6 - (x^2 + y^2))$. Let ω be the differential 1-form in $\mathbf{R}^3$ defined by $\omega = yz^2\, dx + xz\, dy + x^2y^2\, dz$. After choosing an orientation of S, evaluate the integral

$$\int_S z\, dx \wedge dy + d\omega.$$

6. Let $f\colon U \to \mathbf{R}^1$ where $U := (0, 1) \times (0, 1)$. Thus, $f = f(x, y)$ is a function of two variables. Assume for each fixed $x \in (0, 1)$, that $f(x, \cdot)$ is a continuous function of y. Let $\mathcal{F}$ denote the countable family of functions $f(\cdot, r)$ where $r \in (0, 1)$ is a rational number. Thus, for each rational number $r \in (0, 1)$, $f(\cdot, r)$ is a function of x. Assume that the family $\mathcal{F}$ is equicontinuous. Now prove that f is a continuous function of x and y; that is, prove that $f\colon U \to \mathbf{R}^1$ is a continuous function.

7. Let $f_1 \geq f_2 \geq f_3 \geq \dots$ be a sequence of real-valued continuous functions defined on the closed unit ball $B \subset \mathbf{R}^n$ such that $\lim_{k\to\infty} f_k(x) = 0$ for each $x \in B$. Prove that $f_k \to 0$ uniformly on B. This is a special case of Dini's theorem. You may not appeal to Dini's theorem to answer the problem.

8. Let $f\colon \mathbf{R}^1 \to \mathbf{R}^1$ be a nonnegative function satisfying the Lipschitz condition $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$ for all $x_1, x_2 \in \mathbf{R}^1$ and where $K > 0$. Suppose that

$$\int_0^\infty f(x)\,dx < \infty.$$

Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

9. Let F be a nonnegative, continuous real-valued function defined on the infinite strip $\{(x, y) : 0 \leq x \leq 1, y \in \mathbf{R}^1\}$ with the property that $F(x, y) \leq 4$ for all $(x, y) \in [0, 1] \times [0, 2]$. Let f_n be a continuous piecewise-linear function from $[0, 1]$ to $\mathbf{R}^1$ such that $f_n(0) = 0$, f_n is linear on each interval of the form $[\frac{i}{n}, \frac{i+1}{n}]$, $i = 0, 1, \dots, n-1$, and for $x \in (\frac{i}{n}, \frac{i+1}{n})$, $f_n'(x) = F(\frac{i}{n}, f_n(\frac{i}{n}))$. Prove that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that f_{n_k} converges uniformly to a function f on $[0, 1/2]$.