ANALYSIS EXAMINATION

January, 1998

Instructions: Answer all seven questions. Each of the seven questions is equally weighted.
Notation: R denotes the set of all real numbers.

1. State whether each of the following limit exists, and prove your assertions.

   (a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^4+y^4}$

   (b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^4+y^6}$

2. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a uniformly continuous function on $\mathbf{R}$. If

$$f_k(x) = k \int_x^{x+(1/k)} f(t)\,dt \qquad \text{for } x \in \mathbf{R} \text{ and } k = 1,\ 2,\ 3,\ \ldots,$$

prove that the sequence $\{f_k\}$ converges to f uniformly on $\mathbf{R}$.

3. Compute the surface integral $\displaystyle \iint_S (x^2 + y^2)\,dA$, where S is the boundary of the set $\{(x,y,z) \in \mathbf{R}^3 : \sqrt{x^2+y^2} \leq z \leq 1\}$, and dA denotes the surface area element.

4. Let $\pi_1 : \mathbf{R}^2 \to \mathbf{R}$ and $\pi_2 : \mathbf{R}^2 \to \mathbf{R}$ be the projection maps

$$\pi_1(x,y) = x, \quad \pi_2(x,y) = y, \qquad \text{for } (x,y) \in \mathbf{R}^2,$$

and let S be the horizontal strip $S = \{(x,y) \in \mathbf{R}^2 : -1 \leq y \leq 1\}$. State whether each of the following assertions is TRUE or FALSE, and prove your assertions.

   (a) If E is a closed subset of $\mathbf{R}^2$ such that $E \subset S$, then the image $\pi_1(E)$ must be a closed subset of $\mathbf{R}$.

   (b) If E is a closed subset of $\mathbf{R}^2$ such that $E \subset S$, then the image $\pi_2(E)$ must be a closed subset of $\mathbf{R}$.

5. If f is a continuous function on $[0,1]$, prove that

$$\lim_{t\uparrow 1}\left[(1-t)\sum_{k=0}^{\infty} t^k f(t^k)\right] = \int_0^1 f(x)\,dx.$$

6. Let Ω be a bounded, connected open set in $\mathbf{R}^n$, and let f be a continuous real-valued function on the closure of Ω. Suppose that f is of class C^∞ on the set Ω, and suppose that for each point $p \in \Omega$ there is at least one index $i \in \{1,2,\ldots,n\}$ such that

$$\frac{\partial^2 f}{\partial x_i^2}(p) < 0.$$

If

$$f(p) \geq 0 \qquad \text{for every point } p \text{ in the boundary of } \Omega,$$

prove that

$$f(p) \geq 0 \qquad \text{for every point } p \text{ in } \Omega.$$

7. Let Ω be a convex open set in $\mathbf{R}^2$. Let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be functions of class C^∞, and assume that for each point $p \in \Omega$ we have

$$\frac{\partial f}{\partial x}(p) \geq 5, \qquad \frac{\partial g}{\partial y}(p) \geq 5, \qquad \left|\frac{\partial f}{\partial y}(p)\right| \leq 1, \qquad \left|\frac{\partial g}{\partial x}(p)\right| \leq 1.$$

Define $T : \Omega \to \mathbf{R}^2$ by $T(p) = (f(p), g(p))$ for each point $p \in \Omega$. Prove that the image $T(\Omega)$ is an open subset of $\mathbf{R}^2$, and that T is a one-to-one mapping from Ω onto $T(\Omega)$.