Tier 1 Analysis Examination

August 1997

1. Does $a_k = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdots(2k+1)}{2\cdot 4\cdot 6\cdots\cdots 2k}$ converge or diverge? Prove your assertion.

2. Let $S \subset \mathbb{R}^3$ be the "tin can without a lid".

$$\{(x,y,z) : x^2+y^2 = 1,\ 0 \le z \le 1\} \cup \{(x,y,z) : x^2+y^2 \le 1, z = 0\}$$

   Compute the flow $\iint_S F \cdot N\, dA$ "out" of the can if $F = (x(y+z), -zy, -zy)$

3. *Definition* : A transformation of class C^1 $F : \mathbb{R}^3 \to \mathbb{R}^3$ is called volume preserving if for every cube $C \subset \mathbb{R}^3$, with faces parallel to the coordinate planes, volume $(F(C)) =$ volume (C).

   (i) Show that $F(x,y,z) = (x+y, z-4, z^2-y)$ is volume preserving.

   (ii) Show that if $G : \mathbb{R}^3 \to \mathbb{R}^3$ is volume preserving then the determinant of its derivative G' equals ± 1, *and* G maps open sets into open sets.

4. Let $f_n(x) = \int_{1/2}^{x} \arctan^2(t/n)\, dt \quad n = 1, 2, \cdots$

   (i) Show that $\sum_{n=1}^{\infty} f_n'(x)$ (sum of derivatives) is uniformly convergent on $[-1, 1]$.

   (ii) Show that $g(x) = \sum_{n=1}^{\infty} f_n(x)$ is differentiable for all x.

5. Let I_j be a countable family of closed intervals whose interiors are pairwise disjoint and such that $\cup I_j = [0,1]$. Show directly (without fancy integration theorems) that $\sum_{j=1}^{\infty} |I_j| = 1$.

6. Give a counterexample to this statement: Every $f : \mathbb{R} \to \mathbb{R}$ with the property that $f^{-1}(K)$ is compact for any compact K is continuous.

7. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with $0 \in \mathbb{R}^2$ a critical point. Suppose the matrix of second partials $\left(\frac{\partial^2 g}{\partial x_i \partial x_j}(0)\right)$ has eigenvalues -2 and 0. Show that the origin is NOT a local minimum of g.

8. *Definition* : A metric space is said to have property Z if every sequence with exactly one cluster point converges. (Recall that every neighborhood of a cluster point of $\{x_n\}$ contains infinitely many x_n).

   (i) Give an example of a metric space that has property Z and an example of a metric space that does not.

   (ii) What properties of metric spaces are implied by or equivalent to property Z?