

ALGEBRA SYLLABUS

Linear Algebra, including:

- (1) Vector spaces, linear independence, bases, dimension.
- (2) Linear Transformations, matrices, change of basis, eigenvalues and eigenvectors, diagonalization, *Jordan canonical forms*.

Groups, including:

- (1) Subgroups, normal subgroups, quotient groups, and homomorphism theorems
- (2) Standard examples (e.g. cyclic groups, symmetric groups)
- (3) Direct products, structure of finite abelian groups.

Rings, including:

- (1) Ideals, quotient rings, homomorphism theorems.
- (2) UFD's, PID's, Euclidean domains, polynomial rings.
- (3) Maximal ideals, prime ideals, fields, integral domains.
- (4) Field extensions, algebraic elements, finite fields.

References:

Each of the following books has all the material required for the examination.

1. Fraleigh, John B., A First Course in Abstract Algebra, Addison Wesley.
2. Herstein, I.N., Topics in Algebra, Wiley.
3. Beachy, John, and Blair, William, Abstract Algebra with a Concrete Introduction, Prentice Hall.
4. Durbin, John, Modern Algebra - An Introduction, Wiley.

TIER ONE ALGEBRA EXAM

August, 1996

The ring of integers is denoted \mathbb{Z} . The ring of rational numbers is denoted \mathbb{Q} . All rings have identity.

(30)1. Give an example of each of the following (No justification required):

- (a) A nonabelian group of order 18.
- (b) A ring R with exactly four ideals (including R and $\{0\}$).
- (c) A nonabelian group all of whose proper subgroups are cyclic.
- (d) An element in S_{11} of order 21.
- (e) An infinite group all of whose elements are of finite order.
- (f) A noncommutative ring in which every nonzero element is either a zero divisor or a unit.

(15)2. Let H, K be subgroups of a group G with K normal.

- (a) Prove that $HK = \{hk | h \in H, k \in K\}$ is a subgroup of G .
- (b) Provide an example to show that HK need not be a subgroup if neither H nor K is normal.

(15)3. Let G be a group with exactly three subgroups (including $\{e\}$ and G).

- (a) Prove that G is a cyclic group.
- (b) Prove that the order of G is p^2 for some prime p .

(15)4. Let p be a prime and let $R = \{a \in \mathbb{Q} | a = n/m \text{ for } n, m \in \mathbb{Z}, p \nmid m\}$.

- (a) Prove R is a ring under the usual operations in \mathbb{Q} .
- (b) Prove that R contains a unique maximal ideal.

(10)5. Let R be a commutative ring. Let I, J be ideals in R . Prove that the canonical ring homomorphism $\pi : R \rightarrow R/I \oplus R/J$ given by $\pi(r) = (r + I, r + J)$ is an isomorphism if and only if

- (1) $I \cap J = 0$
- and (2) $I + J = R$.

(10)6. An element x in a ring R is called nilpotent if $x \neq 0$ but $x^k = 0$ for some $k > 0$. Find a necessary and sufficient condition on n for the ring $\mathbb{Z}/n\mathbb{Z}$ to contain a nilpotent element. Prove your answer.

(15)7. (a) Prove that if F is a field then F contains a subfield isomorphic to \mathbb{Q} or to $\mathbb{Z}/p\mathbb{Z}$ for some prime p .

(b) Prove that if F is a finite field then there is a prime q such that the number of elements in F is q^k for some positive integer k .

(c) Prove there exists a field containing exactly 7^3 elements.

(10)8. Let R be an integral domain and let S be a subring of R . Prove that if S is a field and R is finite dimensional as an S -vector space, then R is a field.

(20)9. Let A be the following 3×3 matrix over \mathbb{Q} :

$$\begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Find the characteristic polynomial of A .

(b) Find the minimal polynomial of A over \mathbb{Q} .

(c) Let $\mathbb{Q}[A] = \{a_0I + a_1A + \dots + a_kA^k \mid k \geq 0, a_0, a_1, \dots, a_n \in \mathbb{Q}\}$. Prove $\mathbb{Q}[A]$ is a field.

Tier 1 Algebra Examination

August, 1997.

Time: 3 Hours

Question 1 is worth 25 points, questions 2 through 4 are worth 10 points each and questions 5 through 7 are worth 15 points each.

Start each question on a fresh sheet of paper.

1. Give examples (no need to prove anything) or give mathematical reasons if you can not give examples:
 - a. An infinite group all of whose elements have orders 1 or 3.
 - b. Matrices A and B of sizes 3×2 and 2×3 respectively such that $A \cdot B = I_3$,
(identity matrix of size 3×3)
 - c. Two elements in the alternating group A_5 which are conjugate in the symmetric group S_5 but not in A_5 .
 - d. An u.f.d. which is not a p.i.d.
 - e. A transcendental element $\alpha \in \mathbb{C}$ such that $\alpha - \frac{1}{\alpha}$ is an algebraic element.

2.
 - a. Let f be an automorphism of the group $\mathbb{Z}/16\mathbb{Z}$. Show that there exists an odd integer m ($1 \leq m \leq 15$) such that $f(x) = mx$ for all $x \in \mathbb{Z}/16\mathbb{Z}$.
 - b. Decompose the group $\text{Aut}(\mathbb{Z}/16\mathbb{Z})$ of all automorphisms of $\mathbb{Z}/16\mathbb{Z}$ as a product of cyclic groups.

3.
 - a. Let G be a group and H be a subgroup. Let $N = \{ g \in G \mid gHg^{-1} = H \}$ be the normalizer of H in G . Show that there is a bijective correspondence between the left cosets of N in G and the set $\mathcal{S} = \{ K \mid K = xHx^{-1} \text{ for some } x \in G \}$ of all conjugates of H .
 - b. Let G , H , and N be as in (a) above. If in addition, G is finite with $|H| = r$ and $[G:H] = s$, then show that the union of all members of \mathcal{S} (i.e. $\bigcup_{K \in \mathcal{S}} K$) has at most $(rs - s + 1)$ elements.

4. Let V be an n -dimensional vector space.
- Show that a proper subspace W of V is the intersection of all subspaces of V of dimension $n-1$ which contain W .
 - Let $A(V)$ be the vector space of all linear transformations of V to itself. For $x \neq 0$ in V , compute the dimension of $A_x(V) = \{ T \in A(V) \mid T(x) = 0 \}$.
5. Let R be a commutative ring with 1. For an ideal I of R , define \sqrt{I} to be the set $\{ x \in R \mid x^n \in I \text{ for some integer } n \geq 1 \}$.
- Show that \sqrt{I} is an ideal of R which contains I .
 - If I is a prime ideal, show that $\sqrt{I} = I$.
 - If R is a u.f.d and x is a non-zero, non-unit element in R , find a y such that $\sqrt{R \cdot x} = R \cdot y$. (Hint: consider a prime power factorization for x).
6. Let R be an Euclidean domain with a valuation v (i.e. v is a function from the set of non-zero elements of R to the set of non-negative integers such that (i) $v(x) \leq v(xy)$ for $x, y \in R \setminus \{0\}$ and (ii) given $z \in R$ and $y \in R \setminus \{0\}$, there exist q and r such that $z = yq + r$ with $r = 0$ or $v(r) < v(y)$). Assume further that $v^{-1}(n)$ is finite for all n .
- Show that for any non-zero ideal I , R/I is finite. (Note that $I = R \cdot y$ for some y).
 - For the ring $\mathbb{Z}[i] = \{ a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z} \}$ of Gaussian integers with standard valuation v given by $v(a + bi) = a^2 + b^2$, prove that $\mathbb{Z}[i] / 3 \cdot \mathbb{Z}[i]$ is a field of 9 elements. (Hint: show that 3 is a prime).
7. Let p be a prime and n be a positive integer relatively prime to p . Let K be the splitting field of $x^n - 1$ over F_p , the prime field of p elements. Let $[K : F_p] = m$.
- Show that n divides $p^m - 1$. (a hint is given below)
 - If r is such that n divides $p^r - 1$, show that $m \leq r$.
- (Hint for parts (a) and (b): Show first that roots of $x^n - 1$ are all distinct and they form a subgroup of the multiplicative group $K \setminus \{0\}$ which is cyclic).
- Find $[K : F_3]$ where K is the splitting field of $x^{14} - 1$.

Tier 1 Algebra Examination

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(identity matrix of size 3×3)
 - c. Two elements in the alternating group A_5 which are conjugate in the symmetric group S_5 but not in A_5 .
 - d. An u.f.d. which is not a p.i.d.
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 - b. Decompose the group $\text{Aut}(\mathbb{Z}/16\mathbb{Z})$ of all automorphisms of $\mathbb{Z}/16\mathbb{Z}$ as a product of cyclic groups.

3.
 - a. Let G be a group and H be a subgroup. Let $N = \{ g \in G \mid gHg^{-1} = H \}$ be the normalizer of H in G . Show that there is a bijective correspondence between the left cosets of N in G and the set $\mathcal{S} = \{ K \mid K = xHx^{-1} \text{ for some } x \in G \}$ of all conjugates of H .
 - b. Let G , H , and N be as in (a) above. If in addition, G is finite with $|H| = r$ and $[G:H] = s$, then show that the union of all members of \mathcal{S} (i.e. $\bigcup_{K \in \mathcal{S}} K$) has at most $(rs - s + 1)$ elements.

4. Let V be an n -dimensional vector space.
- Show that a proper subspace W of V is the intersection of all subspaces of V of dimension $n-1$ which contain W .
 - Let $A(V)$ be the vector space of all linear transformations of V to itself. For $x \neq 0$ in V , compute the dimension of $A_x(V) = \{ T \in A(V) \mid T(x) = 0 \}$.
5. Let R be a commutative ring with 1. For an ideal I of R , define \sqrt{I} to be the set $\{ x \in R \mid x^n \in I \text{ for some integer } n \geq 1 \}$.
- Show that \sqrt{I} is an ideal of R which contains I .
 - If I is a prime ideal, show that $\sqrt{I} = I$.
 - If R is a u.f.d and x is a non-zero, non-unit element in R , find a y such that $\sqrt{R \cdot x} = R \cdot y$. (Hint: consider a prime power factorization for x).
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- Show that for any non-zero ideal I , R/I is finite. (Note that $I = R \cdot y$ for some y).
 - For the ring $\mathbb{Z}[i] = \{ a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z} \}$ of Gaussian integers with standard valuation v given by $v(a + bi) = a^2 + b^2$, prove that $\mathbb{Z}[i] / 3 \cdot \mathbb{Z}[i]$ is a field of 9 elements. (Hint: show that 3 is a prime).
7. Let p be a prime and n be a positive integer relatively prime to p . Let K be the splitting field of $x^n - 1$ over F_p , the prime field of p elements. Let $[K : F_p] = m$.
- Show that n divides $p^m - 1$. (a hint is given below)
 - If r is such that n divides $p^r - 1$, show that $m \leq r$.
- (Hint for parts (a) and (b): Show first that roots of $x^n - 1$ are all distinct and they form a subgroup of the multiplicative group $K \setminus \{0\}$ which is cyclic).
- Find $[K : F_3]$ where K is the splitting field of $x^{14} - 1$.

Tier 1 Examination - Algebra

January , 1998

Note: (1) Justify your answers.

(2) All rings are assumed to have identity elements.

(3) The ring of integers is denoted \mathbf{Z} ; the real numbers are denoted \mathbf{R} .

(24)1. Prove each of the following statements:

(a) If G and H are groups and $f : G \rightarrow H$ is a group homomorphism, then the kernel of f is a normal subgroup of G .

(b) If N is a normal subgroup of index 12 in a group G and $g \in G$ with $g^5 \in N$, then $g \in N$.

(c) If R is a commutative ring in which the only ideals are 0 and R , then R is a field.

(d) If I and J are ideals in a commutative ring R then the set $S = \{r \in R \mid rI \subseteq J\}$ is an ideal of R .

(8)2. Determine the number of subgroups of the group $C_5 \times C_5$.

(8)3. Determine the units in the polynomial ring $\mathbf{F}_2[x]$, where \mathbf{F}_2 denotes the ring $\mathbf{Z}/2\mathbf{Z}$.

(10)4. Determine whether the following matrix is diagonalizable over \mathbf{R} .

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 5 & 4 \\ -1 & -7 & -5 \end{pmatrix}$$

(10)5. Classify the finite abelian groups with the property that every proper subgroup is cyclic. (A subgroup H of a group G is called proper if $H \neq G$.)

(10)6. Let R be a finite ring, not necessarily commutative. Prove that if $a \in R$ is not invertible then there must exist a nonzero element $b \in R$ such that $ab = 0$.

(10)7. Prove that if R is a commutative ring and the polynomial ring $R[x]$ is a PID, then R must be a field.

(10)8. Let V be a finite dimensional real vector space and let $T : V \rightarrow V$ be a linear transformation such that $T^2 = T$.

(a) Prove that $\ker(T)$ and $T(V)$ are complementary subspaces of V , that is that $\ker(T) \cap T(V) = 0$ and $\ker(T) + T(V) = V$.

(b) Prove there is a basis of V for which the matrix of T has the following form, where I_m is the $m \times m$ identity matrix and $0_{r,t}$ is the $r \times t$ zero matrix.

$$\begin{pmatrix} I_m & 0_{m,s} \\ 0_{s,m} & 0_s \end{pmatrix}$$

(10)9. Let F be a field and let K/F be a field extension of odd degree. Prove that if $K = F(a)$ for some element $a \in K$, then $F(a) = F(a^2)$.

1. Give examples (no need to prove anything): (12 points)
 - a. Two non-isomorphic abelian groups of order 108 such that the order of every element divides 72.
 - b. A linear transformation $T : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ such that the eigenvectors of T do not span \mathbb{C}^3 .
 - c. A unique factorization domain D and a pair of elements $u, v \in D$ such that the greatest common divisor of u and v is NOT a linear combination of u and v .
 - d. Three ring homomorphisms from $\mathbb{Z} \rightarrow \mathbb{Z}_{10}$.
(A ring homomorphism between rings A and A' is a map $f : A \rightarrow A'$ such that for $a, b \in A$, $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$.)

2. Let G be a group. An equivalence relation \equiv on G is called a congruence relation if $g_1 \equiv g_2$ and $h_1 \equiv h_2$ implies $g_1h_1 \equiv g_2h_2$.
 - a. Suppose that \equiv is a congruence relation on G . Show that the equivalence class of the identity element of G is a normal subgroup of G . (5 points)
 - b. Let $\phi : G \rightarrow G'$ be a homomorphism of groups and \equiv' be a congruence relation on G' . Define a relation \equiv on G by: $x \equiv y$ iff $\phi(x) \equiv' \phi(y)$. Show that \equiv is an equivalence relation on G that is also a congruence relation. (4 points)

3. Let A be a commutative ring with 1. An element $x \in A$ is called nilpotent if $x^r = 0$ for some positive integer r .
 - a. Show that the set N of all nilpotent elements in A is an ideal in A and that the quotient ring A/N has no non-zero nilpotent elements. (6 points)
 - b. Show that if $x \in N$, then $1 - x$ is a unit in A . (Hint: Factor $u^r - v^r$ and specialize.) (3 points)

4. Let $\mathbf{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree 2 or less with real coefficients. Consider the linear transformation $T : \mathbf{P}_2(\mathbb{R}) \rightarrow \mathbf{P}_2(\mathbb{R})$ given by:

$$T(f)(x) = f(0) + f(1)(x + x^2).$$

Find the eigenvalues of T and determine whether T is diagonalizable. (8 points)

5. Let G be the subgroup of 2×2 complex invertible matrices generated by $x = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $y = \begin{bmatrix} 0 & \omega \\ -\omega^{-1} & 0 \end{bmatrix}$, where ω is a primitive cube-root of 1. Let H and K be the subgroups of G generated by x and y respectively.

a. Show that H and K are normal in G . (6 points)

b. Compute the orders of the subgroups $H, K, H \cap K$, and G . (4 points)

6. Let V be an n -dimensional real vector space. Consider the set $F(V)$ of all functions from $V \times V$ to \mathbb{R} . It can be proved that $F(V)$ is a real vector space under the operations: For all $\phi, \psi \in F(V)$ and $r \in \mathbb{R}$,

$$(i) (\phi + \psi)(x, y) = \phi(x, y) + \psi(x, y), \quad (ii) (r\phi)(x, y) = r\phi(x, y).$$

Let $S(V)$ be the subset of all functions $\phi \in F(V)$ which satisfy the following condition: For all $x, y, x' \in V$ and $a, a' \in \mathbb{R}$,

$$(i) \phi(x, y) = \phi(y, x), \quad (ii) \phi(ax + a'x', y) = a\phi(x, y) + a'\phi(x', y).$$

a. Show that $S(V)$ is a subspace of $F(V)$. (3 points)

b. Find the dimension of $S(V)$. (Hint: Use a suitable map from $S(V)$ to the vector space of all $n \times n$ matrices.) (5 points)

7. Consider the ring $\mathbb{Z}_2[x]$ and two ideals I and J generated by the elements $(x^2 - 1)$ and $x^2 + x + 1$ respectively.
- Find all the units in the quotient rings $\mathbb{Z}_2[x]/I$ and $\mathbb{Z}_2[x]/J$. (7 points)
 - If F is a field of 4 elements, is it true that F is isomorphic to one of these two rings? (Justify your answer.) (3 points)
8. Find the irreducible polynomial over \mathbb{Q} of the element $\alpha = \sqrt{5} \cdot \sqrt[3]{2}$. (Hint: Prove first that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5}, \sqrt[3]{2})$.) (9 points)
9. Let R be the ring $\mathbb{Z}[i]$ of Gaussian integers (i.e. $R = \{a + bi \mid a, b \in \mathbb{Z}\}$.)
- Let $p \in \mathbb{Z}$ be a prime integer. Show that p is a prime element of R if the equation $x^2 + y^2 = p$ has no integer solutions for x and y . (Hint: use the norm $N(a + bi) = a^2 + b^2$.) (5 points)
 - Using (a) or otherwise, show that 11 does not divide $4n^2 + 1$ for all $n \in \mathbb{Z}$. (5 points)
10. Give reasons why the following examples *do not* exist: (15 points)
- Elements $x, y \in S_5$ of order 3 and 4 respectively such that $xy = yx$.
 - Elements $\alpha, \beta \in \mathbb{C}$ with α transcendental over \mathbb{Q} and $\beta, \alpha^2 + \beta$ both algebraic over \mathbb{Q} .
 - An integral domain with 20 elements.

Tier One Algebra Exam

January 1999

4 Hours

Each problem is 10 points.

1. Find an invertible matrix M such that $M^{-1}AM$ is diagonal, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. Give an example of a 2×2 matrix that does not have two linearly independent eigenvectors.
3. A square matrix A is called nilpotent if $A^k = 0$ for some $k > 0$.
- a) Give an example with justification of a nonzero nilpotent A .
- b) Prove that if A is nilpotent, then $I + A$ is invertible.
4. Let G be a finite group, $a, b \in G$. Prove that the orders of ab and ba are equal.
5. Prove that the set of elements of finite order in an abelian group is a subgroup.
6. Let G be a finite group of order > 2 . Prove that G has a nontrivial automorphism.
7. Find two generators for the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by $\{(8, 7), (2, 5), (9, 3)\}$.
8. Consider the ring homomorphism

$$f : \mathbb{Z}[x] \rightarrow \mathbb{R}$$

which maps x to $\sqrt[3]{2}$. Consider the ideal $\text{Ker}(f)$ in $\mathbb{Z}[x]$. Show $\text{Ker}(f)$ is generated by a single polynomial and find that polynomial.

9. Prove that the ring $H := \mathbb{F}_2[x]/(x^3 + x^2 + 1)$ is a field. Find the degree of the field extension $[H : \mathbb{F}_2]$.

10. Let R be a commutative ring and $I \subset R$ be an ideal. Consider the set

$$J := \{x \in R \mid x^n \in I \text{ for some } n \geq 1\}.$$

a) Show that J is an ideal in R .

An ideal I is called **primary** if for all x and y satisfying $xy \in I$, either $x \in I$ or $y^m \in I$ for some $m \geq 1$, where m may depend on y .

b) Show that if I is primary, J is prime.

11. Show that if some element of a commutative ring has three or more square roots, the ring is not an integral domain.

Tier 1 Examination - Algebra

August 27, 1999

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by \mathbf{R} .

(8)1. Let G be a group and let H, K be subgroups of G , with H normal. Prove that $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G .

(8)2. a. Find the eigenvalues of the following real matrix.

$$\begin{pmatrix} 0 & 3 & 2 \\ 0 & -1 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

b. Determine whether or not the matrix is diagonalizable over the reals.

(12)3. Let G be a group, H a subgroup of G .

a. Prove or disprove: If the index $[G : H] = 2$, then H is normal.

b. Prove or disprove: If the index $[G : H] = 3$, then H is normal.

(8)4. A ring R is called simple if its only (two-sided) ideals are $\{0\}$ and R . Prove that if R is simple then the center of $R (= \{x \in R \mid xr = rx \text{ for all } r \in R\})$ is a field.

(8)5. Prove that if A and B are real $n \times n$ matrices, then AB and BA have the same eigenvalues.

(7)6. Let p be a prime number and let F be a field of characteristic p . Prove that if $a \in F$ satisfies $a^p = 1$, then $a = 1$.

(8)7. Prove that if R is a principal ideal domain, then every nonzero prime ideal is a maximal ideal. Does the conclusion hold in general for R a unique factorization domain?

(8)8. Let F be a finite field. Prove there is a prime p such that the number of elements in F is p^k for some positive integer k .

(15)9. Let V be an n -dimensional real vector space. Let $A(V)$ be the set of linear transformations on V , that is $A(V) = \{T : V \rightarrow V \mid T \text{ is a linear transformation}\}$. Recall that $A(V)$ is also a real vector space, if we define the sum by $(S + T)(v) = S(v) + T(v)$ for $S, T \in A(V)$ and $v \in V$ and the scalar product by $(aT)(v) = aT(v)$ for $a \in \mathbb{R}$ and $T \in A(V)$.

a. What is the dimension of $A(V)$?

b. Let $B(V) = \{T \in A(V) \mid \dim T(V) < n\}$. Determine whether or not $B(V)$ is a subspace of $A(V)$. If it is, find its dimension.

c. Let W be a subspace of V of dimension k and let $C(V) = \{T \in A(V) \mid T(w) = 0 \text{ for all } w \in W\}$. Determine whether or not $C(V)$ is a subspace of $A(V)$. If it is, find its dimension.

(10)10. Let $G = Z_3 \times Z_3$, where Z_3 denotes the cyclic group of order 3.

a. Determine the number of distinct homomorphisms from G to itself.

b. Determine the number of distinct isomorphisms from G to itself.

(8)11. Let R be an integral domain and let S be a subring of R . Assume S is a field. Then we may view R as an S -vector space in a natural way. Prove that if R is finite dimensional as an S -vector space, then R is a field.

Algebra Tier 1 Examination

January 2000

- 1a. Let G be an abelian group of order 60. Consider the homomorphism $\phi : G \rightarrow G$ given by $\phi(g) = g^5$. Let $H = \text{Ker}(\phi)$ and $K = \text{Image}(\phi)$. Show that G is the internal direct product of H and K . (Hint: g.c.d. of integers is a linear combination). (8 points)
- 1b. Would the conclusion in (1a) necessarily be valid for the homomorphism η given by $\eta(g) = g^{10}$? (4 points)
2. A subgroup K of a group G is called a characteristic subgroup if for any automorphism θ of G , $\theta(K) = K$.
- 2a. Show that all subgroups of a cyclic group are characteristic. (5 points)
- 2b. Show that the center of a group is a characteristic subgroup. (5 points)
- 2c. If K is a normal subgroup of G and H is a characteristic subgroup of K show that H is a normal subgroup of G . (7 points)
- 2d. Consider the alternating group A_4 . Give example of a characteristic subgroup K (of A_4) and a normal subgroup H of K such that H is not a normal subgroup of A_4 . (5 points)
- 3a. Let K be an extension of a field F . If $\alpha \in K$ is transcendental over F , show that so is $\beta = \alpha^2 + \frac{1}{\alpha^2}$. (5 points)
- 3b. Let K be an extension of F of degree n . Let f be an irreducible polynomial in $F[x]$ of degree m . If the g.c.d. of m and n is 1, show that f remains irreducible when considered as a polynomial in $K[x]$. (Hint: consider a root α of f in an algebraic closure \overline{F} of F which contains K .) (7 points)
- 4a. Is it possible to have a finite field which is algebraically closed? Justify your answer. (4 points)
- 4b. Let E be an extension of $\overline{\mathbb{Z}_p}$ contained in an algebraic closure $\overline{\mathbb{Z}_p}$. Let f be an irreducible polynomial in $\overline{\mathbb{Z}_p}[x]$ and let $\alpha, \beta \in \overline{\mathbb{Z}_p}$ be roots of f . If $\alpha \in E$, show that $\beta \in E$. (6 points)

5. For a ring R with unity 1, an element $r \in R$ is said to be a unit if there exists an element $s \in R$ such that $rs = 1 = sr$.

5a. Find all the units of the ring $\mathbb{Z}[i] = \{m + ni \in \mathbb{C} \mid m, n \in \mathbb{Z}\}$. (Hint: think of modulus of a complex number) (5 points)

5b. Give example of a ring with exactly 20 units. (5 points)

5c. Let $\mathbb{C}[[x]]$ be the ring of formal power series, i.e.

$\mathbb{C}[[x]] = \{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in \mathbb{C}\}$ with addition and multiplication given by

$$(\sum_{i=0}^{\infty} a_i x^i) + (\sum_{i=0}^{\infty} b_i x^i) = \sum_{i=0}^{\infty} (a_i + b_i) x^i \text{ and}$$

$$(\sum_{i=0}^{\infty} a_i x^i) \cdot (\sum_{i=0}^{\infty} b_i x^i) = \sum_{i=0}^{\infty} c_i x^i, \text{ where } c_i = \sum_{j+k=i} a_j b_k.$$

Show that an element $r = \sum_{i=0}^{\infty} a_i x^i$ is a unit in $\mathbb{C}[[x]]$ if and only if $a_0 \neq 0$. (5 points)

5d. Let I be a non-zero ideal in $\mathbb{C}[[x]]$. Show that there exists a positive integer k such that $x^k \in I$. Show further that I is a principal ideal. (7 points)

6a. Let A be an $n \times m$ matrix and let B be $m \times n$ matrix with real coefficients such that $A \cdot B = I_n$, the identity matrix of size n . What is the relationship between n and m ? (Justify your answer). Further, if $n = m$, show that $B \cdot A = I_n$ as well. (Hint: think of corresponding linear transformations of \mathbb{R}^n). (7 points)

6b. Let T be a linear transformation of a finite dimensional vector space over \mathbb{R} . Let V_1 (respectively V_{-1}) denote the eigenspace of T for the eigenvalue 1 (respectively -1). If $T^2 = Id$, show that V is a direct sum of V_1 and V_{-1} . (Hint: think of $v + T(v)$). (7 points)

6c. Give an example of a 4×4 matrix with real entries whose real eigenvalues are ± 1 and whose complex eigenvalues are $\pm i$ and $\pm i$ (no need to justify). (3 points)

6d. Let V be a finite dimensional vector space over \mathbb{R} and W, W' be two subspaces of V . Show that $\dim(W + W') = \dim(W) + \dim(W') - \dim(W \cap W')$. (5 points)

Tier One Algebra Exam

August, 2000

1. (14 points) Let A be the 3×3 matrix all of whose entries are 1, i.e.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

- (i) Find the characteristic polynomial of A .
- (ii) Find the minimal polynomial of A .
- (iii) Find all eigenvalues of A .
- (iv) Is A diagonalizable? If the answer is yes, find P such that PAP^{-1} is diagonal.

If the answer is no, provide a reason.

2. (8 points) Let T and S be linear transformations from R^n to R^m . The coincidence set for T and S is defined to be the set $C(T, S) = \{w \in R^m \mid T(x) = w = S(y), \text{ for some } x, y \in R^n\}$. Let T and S be the linear transformations from R^4 to R^3 represented by the 3×4 matrices

$$T = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 5 & 4 & 1 & 0 \\ 3 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 2 \\ -1 & 4 & 11 & 6 \end{pmatrix}$$

Find a basis for $C(T, S)$.

3. (12 points) Let G be a group.

(i) Prove that the intersection of a family $F = \{H_t : t \in T\}$ of subgroups of G is a subgroup.

(ii) Let H be a subgroup of G . Prove that the intersection $K = \cap \{gHg^{-1} : g \in G\}$ of all conjugates of H is a normal subgroup.

(iii) Let H and K be as in (ii). If $[G : H] < \infty$, prove that $[G : K] < \infty$.

4. (6 points) Give an example of a nonabelian group with every proper subgroup abelian.

5. (6 points) Let S_{10} denote the symmetric group in 10 letters. Find the smallest n such that $a^n = 1$, for all $a \in S_{10}$. Justify your answer.

6. (10 points)

(i) Let G be an abelian group of order mn , where m and n are relatively prime. Let $H := \{g \in G : |g| \text{ divides } m\}$ and $K := \{g \in G : |g| \text{ divides } n\}$. Prove that the homomorphism $H \times K \rightarrow G$ induced by inclusion is an isomorphism.

(ii) Let G be an abelian group of order $p_1^{e_1} \cdots p_t^{e_t}$, where p_i 's are distinct primes. For each prime p dividing $|G|$, let $G_p := \{g \in G : |g| = p^n \text{ for some } n\}$. Prove that $G \cong G_{p_1} \times \cdots \times G_{p_t}$.

7. (12 points)

(i) Let p and q be distinct primes. Let $n = pq$. Prove that $\mathbb{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}_q$ as rings.

(ii) Let n, p, q be as in (i). Let ℓ be any integer. Prove that $a^{1+\ell(p-1)(q-1)} \equiv a \pmod{n}$ for all $a \in \mathbb{Z}$.

8. (4 points) Give three examples of nonisomorphic rings of order 4.

9. (10 points)

(i) Prove: If p is prime, $f(x) \in \mathbb{Z}_p[x]$ is a polynomial, and $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

(ii) Does this remain true if p is not prime? Explain.

10. (10 points)

(i) Prove that $x^2 - 3$ and $x^5 - 2$ are irreducible in $\mathbb{Q}[x]$.

(ii) Prove that $x^5 - 2$ is irreducible in $\mathbb{Q}(\sqrt{3})[x]$.

11. (8 points) Write a one or two paragraph essay explaining (without proofs) why trisecting the angle $\pi/3$ is impossible using a straightedge and a compass.

Tier 1 Examination - Algebra

January 4, 2001

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by \mathbf{R} . The set of integers modulo n is denoted \mathbf{Z}_n . The group of permutations on n letters is denoted S_n .

- (20)1. Give an example of each of the following. No justification is required.
- (a) A nonabelian group of order 18.
 - (b) An infinite commutative ring R such that for all $y \in R$, $y + y + y = 0$.
 - (c) A 3 by 3 real matrix that is diagonalizable over the complex numbers but not over the reals.
 - (d) A unique factorization domain that is not a principal ideal domain.
 - (e) An element of order 3 in $\mathbf{GL}_2(\mathbf{R})$.
- (10) 2. Let G be a group with the property that $g^2 = e$ for all $g \in G$. Prove G is abelian.
- (10)3. Determine the number of homomorphisms from S_3 to $\mathbf{Z}_2 \times \mathbf{Z}_4$.
- (10)4. Find $\lim_{n \rightarrow \infty} \begin{pmatrix} 2 & 3 \\ -1/2 & -1/2 \end{pmatrix}^n$.
- (10)5. Let n be a positive odd integer and let $A, B \in M_n(\mathbf{R})$ such that $A^2 = B^2 = I$. Prove that A and B have a common eigenvector (not necessarily with the same eigenvalue).
- (10)6. Let R and S be commutative rings and let $\phi : R \rightarrow S$ be a ring homomorphism. Suppose there is an ideal I of R such that $\ker(\phi) \subset I \subset R$ (proper containments). Prove that the image of ϕ is not a field.
- (10)7. Let F be a subfield of \mathbf{R} and suppose m and n are positive integers with $\sqrt{m} + \sqrt{n} \in F$. Prove that \sqrt{m} and \sqrt{n} are in F .
- (10)8. Let K be a field and let K^\times denote the group of nonzero elements of K . Prove that K^\times contains at most two elements of order 6.

(10)9. Let $G = (\mathbb{Q}, +)/(\mathbb{Z}, +)$, where $(\mathbb{Q}, +)$ denotes the group of rational numbers under addition and $(\mathbb{Z}, +)$ denotes the subgroup of integers under addition. Prove that G is an infinite group in which every element has finite order.

(15)10. (a) Let R be a commutative ring and suppose I and J are ideals of R such that $I + J = R$, where $I + J = \{i + j \mid i \in I, j \in J\}$. Prove the map $\phi : R/I \times R/J \rightarrow R/(I \cap J)$ given by $\phi(r + I, r + J) = (r + I \cap J)$ is an isomorphism of rings.

(b) Let R be a commutative ring containing exactly 4 ideals (including $\{0\}$ and R). Let I and J denote the other two ideals and suppose they are incomparable, that is $I \not\subseteq J$ and $J \not\subseteq I$. Prove that R is isomorphic to the direct product of two fields.

Algebra Tier 1 Examination

August 2001

Time: 4 hours

Notation:

\mathfrak{R} : field of real numbers \mathfrak{R}^n : Euclidean n-space
 \mathbb{Z}_n : ring of integers modulo n

\mathbb{Z} : ring of integers
 \mathbb{Q} : field of rational numbers

1. Give examples (no need to prove anything) OR give mathematical reasons if you can't give examples of the following: (20 points)
 - (a) An integral domain which is not a U.F.D. (unique factorization domain)
 - (b) A field with 6 elements.
 - (c) A 5×3 matrix P and a 3×5 matrix Q (with entries in \mathfrak{R}) such that $P \cdot Q = I_5$ (identity matrix of size 5).
 - (d) A prime number p such that $x^2 - x + 2$ divides $3x^3 - 7x^2 + 40x + 27$ when considered as elements of $\mathbb{Z}_p[x]$.
 - (e) An element of order 15 in S_9 .

- 2a. Let G be a finite group of order n . Let H be a unique subgroup of G of index h . Show that H is a normal subgroup. Give example of a finite group L and at least two of its normal subgroups M and N of the same index. (5 points)

- 2b. Find all elements in \mathbb{Z}_7 which are squares of other elements. Use this and the natural homomorphism $f : \mathbb{Z} \rightarrow \mathbb{Z}_7$ to show that the arithmetic sequence 10, 17, 24, 31 has no perfect squares. (5 points)

- 2c. Describe all the abelian groups G of order 500 such that $g^{50} = e$ (identity of G) for all $g \in G$. (Show all the work.) (5 points)

- 3a. Let H and K be cyclic groups of order 6 and 4 respectively with generators a and b respectively. Let $G = H \times K$. Let L be the subgroup of G generated by (a^5, b^2) . Find the order of the element $\alpha = (a, b) \cdot L$ in the factor group G/L . Hence or otherwise conclude that G/L is cyclic. (8 points)

- 3b. Let G be a group of order 21 which contains unique subgroups of order 3 and 7. Find the number of elements of order 21 in G . (7 points)
- 4a. Let R be a commutative ring with 1. For any $x \in R$, let $I_x = \{r \in R \mid rx = 0\}$. Show that I_x is an ideal in R . Find this ideal for $R = \mathbb{Z}_{24}$ and $x = 15$. (5 points)
- 4b. Let R be a commutative ring with 1 which has exactly 3 ideals $\{0\}, J$ and R . If $a \in R$ is not a zero divisor then show that a is a unit. (Hint: If a is not a unit then show first that $R \cdot a = R \cdot a^2 = J$ and then get a contradiction. (8 points)
- 5a. Show that $x^2 - 4x - 4$ is irreducible in $\mathbb{Z}_5[x]$. (2 points)
- 5b. Show that $R = \mathbb{Z}_5[x]/I$ (where I is the ideal generated by $x^2 - 4x - 4$) is a field. (You may quote and use any relevant results). Find also the number of elements in R . (5 points)
- 5c. Let I, R be as in {5b}. Let $\alpha = x + I \in R$. For this element, find the additive order (i.e. order as an element of the additive group $(R, +)$) and the multiplicative order (i.e order as an element of the multiplicative group $(R - \{0\}, \cdot)$). (5 points)
- 5d. Find elements $a, b \in \mathbb{Z}_5$ such that $(2\alpha + 1)(a\alpha + b) = 1$. (5 points)
- 6a. Let V be a real vector space and let $S = \{v_1, v_2, \dots, v_m\}$ be a maximal linearly independent set (i.e S is not properly contained in another linearly independent set. Show that S is a basis of V . (4 points)
- 6b. Let $V = \mathbb{R}^3$. Let $W = \{(a, b, c) \in \mathbb{R}^3 \mid a = 2b + 3c \text{ and } b^2 = ac\}$. Determine with justification if W is a subspace of V . (4 points)
7. Let V denote the vector space of polynomials of degree at most 1 with real coefficients. Consider the linear transformation $T : V \rightarrow V$ given by $T(f(x)) = -f(1) + f(-3)x$ (where $f(a)$ is the value of the polynomial $f(x)$ at a).
- (a) Find the matrix of T with respect to the basis $\{1, x\}$ of V . (3 points)
- (b) Find the eigenvalues of T and a basis for each eigenspace. (5 points)
- (c) Is T diagonalizable? Justify your answer. (4 points)

Tier 1 Examination - Algebra

January 3, 2002

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by \mathbf{R} and the set of rational numbers by \mathbf{Q} . The set of integers modulo n is denoted \mathbf{Z}_n . The order of a set S is denoted $|S|$.

(10)1. What is the order of the group of invertible 2×2 matrices with entries in \mathbf{Z}_5 ?

(10) 2. Let G and H be finite abelian groups of the same order 2^n . Prove that if for each integer m ,

$$|\{x \in G \mid x^{2^m} = 1\}| = |\{x \in H \mid x^{2^m} = 1\}|,$$

then G and H are isomorphic.

(10)3. Prove or give a counterexample for each of the following:

(a) For every integer n there is a finite group that cannot be generated by n elements.

(b) If G and H are finite groups such that $|G|$ and $|H|$ are relatively prime, then there exists a unique homomorphism from G to H .

(c) Every quotient group H of a group G is isomorphic to a subgroup of G .

(10)4. Prove that there exist algebraic numbers α and β , each of degree 3, such that $[\mathbf{Q}(\alpha, \beta) : \mathbf{Q}] = 6$.

(10)5. Show that for every prime p there are exactly $\frac{p^3-p}{3}$ irreducible cubic polynomials with leading coefficient 1 over the field \mathbf{Z}_p .

(10)6. Prove or disprove: Every maximal ideal of the real polynomial ring $\mathbf{R}[x, y]$ is of the form $(x - a, y - b)$ for some $a, b \in \mathbf{R}$.

(10)7. Prove that for every positive integer $n \geq 6$ which is not prime, there exist integers a and b such that the congruence equation $x^2 + ax + b \equiv 0 \pmod{n}$ has more than two solutions modulo n .

(10)8. Let V be a finite dimensional complex vector space and $T : V \rightarrow V$ a linear transformation. Suppose there exists $v \in V$ such that $\{v, T(v), T^2(v), \dots, T^{n-1}(v)\}$ is a basis for V . Show that the eigenspaces of T are all 1-dimensional.

(10)9. Prove that a 5×5 skew-symmetric matrix A has determinant 0. (Recall that a matrix is called skew-symmetric if the transpose of A is the negative of A .)

(10)10. For each of the following conditions on a complex square matrix M , determine whether the condition implies that M is diagonalizable.

(a) $M^2 = M$

(b) $M^3 = I$

(c) $M^4 = 0$

TIER ONE ALGEBRA EXAM

- (1) Consider the matrix $A = \begin{pmatrix} -4 & 18 \\ -3 & 11 \end{pmatrix}$.
- (a) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.
 - (b) Using the previous part of this problem, find a formula for A^n where A^n is the result of multiplying A by itself n times.
 - (c) Consider the sequences of numbers

$$a_0 = 1, b_0 = 0, a_{n+1} = -4a_n + 18b_n, b_{n+1} = -3a_n + 11b_n.$$

Use the previous parts of this problem to compute closed formulae for the numbers a_n and b_n .

- (2) Let P_2 be the vector space of polynomials with real coefficients and having degree less than or equal to 2. Define $D : P_2 \rightarrow P_2$ by $D(f) = f'$, that is, D is the linear transformation given by taking the derivative of the polynomial f . (You needn't verify that D is a linear transformation.)
- (3) Give an example of each of the following. (No justification required.)
- (a) A group G , a normal subgroup H of G , and a normal subgroup K of H such that K is not normal in G .
 - (b) A non-trivial perfect group. (Recall that a group is *perfect* if it has no non-trivial abelian quotient groups.)
 - (c) A field which is a three dimensional vector space over the field of rational numbers, \mathbb{Q} .
 - (d) A group with the property that the subset of elements of finite order is not a subgroup.
 - (e) A prime ideal of $\mathbb{Z} \times \mathbb{Z}$ which is not maximal.

- (4) Show that any field with four elements is isomorphic to

$$\frac{\mathbb{F}_2[t]}{(1 + t + t^2)}.$$

- (5) Let \mathbb{F}_p denote a field with p elements, p prime. Consider the ring

$$R = \frac{\mathbb{F}_p[x, y]}{\langle x^2 - 3, y^2 - 5 \rangle}$$

where $\langle x^2 - 3, y^2 - 5 \rangle$ denotes the ideal generated by $x^2 - 3$ and $y^2 - 5$. Show that R is not a field.

- (6) Let

$$R = \frac{\mathbb{C}[x, y]}{(x^2 + y^3)}$$

where $\mathbb{C}[x, y]$ is the polynomial ring over the complex numbers \mathbb{C} with indeterminates x and y . Similarly, let S be the subring of $\mathbb{C}[t]$ given by $\mathbb{C}[t^2, t^3]$.

- (a) Prove that R and S are isomorphic as rings.
 - (b) Let I be the ideal in R given by the residue classes of x and y . Prove that I is a prime ideal of R but not a principle ideal of R .
- (7) Suppose that $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map.
- (a) Suppose $n = 2$ and $T^2 = -I$. Prove that T has no eigenvectors in \mathbb{R}^2 .
 - (b) Suppose $n = 2$ and $T^2 = I$. Prove that \mathbb{R}^2 has a basis consisting of eigenvectors of T .
 - (c) Suppose $n = 3$. Prove that T has an eigenvector in \mathbb{R}^3 . Give an example of an operator T such that T has an eigenvector in \mathbb{R}^3 , but \mathbb{R}^3 does not have a basis consisting of eigenvectors of T .
- (8) Let $p(x)$ and $q(x)$ be polynomials with rational coefficients such that $p(x)$ is irreducible over the field of rational numbers \mathbb{Q} . Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ be the complex roots of p , and suppose that $q(\alpha_1) = \alpha_2$. Prove that
- $$q(\alpha_i) \in \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$
- for all $i \in \{2, 3, \dots, n\}$.
- (9) Let F be a field containing subfields F_{16} and F_{64} with 16 and 64 elements respectively. Find (with proof) the order of $F_{16} \cap F_{64}$.
- (10) Let G be a finite group and suppose H is a subgroup of G having index n . Show there is a normal subgroup K of G with $K \subset H$ and such that the order of K divides $n!$.
- (a) Find a matrix representing the linear function D in the basis $\{1, x, x^2\}$.
 - (b) Determine the eigenvalues and eigenvectors of D .
 - (c) Determine if P_2 has a basis such that D is represented by a diagonal matrix. Why or why not?
- (11) Suppose that W is a non-zero finite dimensional vector space over \mathbb{R} . Let T be a linear transformation of W to itself. Prove that there is a subspace U of W of dimension 1 or 2 such that $T(U) \subset U$ (i.e. U is an invariant subspace. Here $T(U)$ denotes the set $\{T(u) | u \in U\}$.)

Tier 1 Algebra Exam

January 2003

1. Give an example of each of the following. (No proofs required.)
 - (a) A square matrix with real coefficients which is not diagonalizable.
 - (b) A linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ whose minimal polynomial is $x^2 - 2x + 1$.
 - (c) An element of the alternating group A_{10} of order 12.
 - (d) Four nonisomorphic groups of order 66.
 - (e) A commutative ring with a prime ideal which is not maximal.
2. Suppose p_1, p_2, p_3 , and p_4 are real polynomials of degree 3 or less. Determine whether either of the following two conditions implies that the set $\{p_1, p_2, p_3, p_4\}$ is linearly dependent in the vector space of real polynomials. In each case, offer a proof or a counterexample.
 - (a) $p_i(0) = 1$ for all i .
 - (b) $p_i(1) = 0$ for all i .
3. Let $T : U \rightarrow V$ be a linear transformation of vector spaces and let $W \subset V$ be a subspace. Prove that $T^{-1}(W)$ is a subspace of U and that
$$\dim(T^{-1}(W)) \geq \dim(U) - \dim(V) + \dim(W).$$
4. Construct a field F with 16 elements as a quotient of $\mathbb{F}_2[x]$ and find a polynomial in $\mathbb{F}_2[x]$ whose image $\alpha \in F$ satisfies $\alpha^3 = 1$ and $\alpha \neq 1$. Justify your answer.
5. Let R be an infinite ring and let $\varphi : M_2(\mathbb{Z}) \rightarrow R$ be a surjective ring homomorphism. Show φ is an isomorphism.
6. Let H and K be subgroups of a group G . Show that $HK = \{hk : h \in H, k \in K\}$ is a subgroup if and only if $HK = KH$.
7. For each $(x, y) \in \mathbb{Z}^2$, let $\langle(x, y)\rangle$ denote the subgroup of \mathbb{Z}^2 generated by (x, y) . Express $\mathbb{Z}^2 / \langle(x, y)\rangle$ as a direct sum of cyclic groups.
8. Let $\alpha \in \mathbb{C}$ be an algebraic number, i.e. a root of a nonzero polynomial with rational coefficients.

- (a) Show that α is a root of a unique monic irreducible polynomial $f(x)$ with rational coefficients.
- (b) Show $\mathbb{Q}[\alpha]$ is a field.
- (c) Show that the degree of $f(x)$ equals $[\mathbb{Q}[\alpha] : \mathbb{Q}]$.
- (d) Write down $f(x)$ when $\alpha = \sqrt{2} + \sqrt{3}$. Justify your answer.
9. Let $F[x]$ denote the polynomial ring over a field F . Let R denote the subring $F[x^2, x^3]$.
- (a) Show $R \subsetneq F[x]$.
- (b) Show that the quotient field of R is $F(x)$.
- (c) Show that R is not a unique factorization domain.
10. For a 2×2 matrix A with real coefficients, define e^A to be the matrix

$$\sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

Assuming that above series converges and that for all 2×2 matrices B and C

$$Be^AC = \sum_{n=0}^{\infty} \frac{1}{n!} BA^nC,$$

compute e^A where $A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$.

TIER ONE ALGEBRA EXAM

- (1) Give an example of each of the following. (No justification required.)
- (a) A group G , a normal subgroup H of G , and a normal subgroup K of H such that K is not normal in G .
 - (b) A non-trivial perfect group. (Recall that a group is perfect if it's only it has no non-trivial abelian quotient groups.)
 - (c) A 2×2 matrix A over \mathbb{R} which is not diagonalizable over \mathbb{R} .
 - (d) A field which is a three dimensional vector space over the field of rational numbers, \mathbb{Q} .
 - (e) A group with the property that the subset of elements of finite order is not a subgroup.
 - (f) A prime ideal of $\mathbb{Z} \times \mathbb{Z}$ which is not maximal.

- (2) A element a in a ring R is called *idempotent* if $a^2 = a$. Show the the only idempotent elements in an integral domain are 0 and 1.

- (3) Consider the matrix $A = \begin{pmatrix} -4 & 18 \\ -3 & 11 \end{pmatrix}$.

- (a) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.
- (b) Using the previous part of this problem, find a formula for A^n where A^n is the result of multiplying A by itself n times.
- (c) Consider the sequences of numbers

$$a_0 = 1, b_0 = 0, a_{n+1} = -4a_n + 18b_n, b_{n+1} = -3a_n + 11b_n.$$

Use the previous parts of this problem to compute a closed formulae for the numbers a_n and b_n .

- (4) Let

$$R = \frac{\mathbb{C}[x, y]}{(x^2 + y^3)}$$

where $\mathbb{C}[x, y]$ is the polynomial ring over the complex numbers \mathbb{C} with indeterminates x and y . Similarly, let S be the subring of $\mathbb{C}[t]$ given by $\mathbb{C}[t^2, t^3]$.

- (a) Prove that R and S are isomorphic as rings.
- (b) Let I be the ideal in R given by the residue classes of x and y . Prove that I is a prime ideal of R but not a principle ideal of R .

- (5) Suppose that $T : R^n \rightarrow R^n$ is a linear map.
- Suppose $n = 2$ and $T^2 = -I$. Prove that T has no eigenvectors in R^2 .
 - Suppose $n = 2$ and $T^2 = I$. Prove that R^2 has a basis consisting of eigenvectors of T .
 - Suppose $n = 3$. Prove that T has an eigenvector in R^3 . **(I suggest we omit this last part to shorten the problem: Give an example of an operator T such that T has an eigenvector in R^3 , but R^3 does not have a basis consisting of eigenvectors of T .)**
- (6) Suppose that W is a non-zero finite dimensional vector space over \mathbb{R} . Let T be a linear transformation of W to itself. Prove that there is a subspace U of W of dimension 1 or 2 such that $T(U) \subset U$ (i.e. U is an invariant subspace. Here $T(U)$ denotes the set $\{T(u)|u \in U\}$.)
- (7) Let $p(x)$ and $q(x)$ be polynomials with rational coefficients such that $p(x)$ is irreducible over the field of rational numbers \mathbb{Q} . Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ be the complex roots of p , and suppose that $q(\alpha_1) = \alpha_2$. Prove that
- $$q(\alpha_i) \in \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$
- for all $i \in \{2, 3, \dots, n\}$.
- (8) Let F be a field containing subfields F_{16} and F_{64} with 16 and 64 elements respectively. Find (with proof) the order of $F_{16} \cap F_{64}$.
- (9) Let G be a finite group and suppose H is a subgroup of G having index n . Show there is a normal subgroup K of G with $K \subset H$ and such that the order of K divides $n!$.
- (10) Let P_2 be the vector space of degree less than or equal to 2 polynomials with real coefficients. Define $D : P_2 \rightarrow P_2$ by $D(f) = f'$, that is, D is the linear transformation given by taking the derivative of the polynomial f . (You needn't verify that D is a linear transformation.)

Find a matrix representing the linear function D in the basis $\{1, x, x^2\}$. Determine the eigenvalues and eigenvectors of D . Determine if P_2 has a basis such that D is represented by a diagonal matrix. Why or why not?

ALGEBRA TIER I EXAM

January 2004

You have 25 questions. Each one is worth 4 points.

1. Let $V \neq \{0\}$ be a vector space over the real numbers \mathbb{R} with the dimension $\dim(V) = n < \infty$. Let $f : V \rightarrow V$ be a linear mapping different the identical mapping id_V and different from $-\text{id}_V$. Assume that f is an involution, i.e., $f \circ f = \text{id}_V$.

- Prove that the only possible eigenvalues are $\lambda = 1$ and $\lambda = -1$.
- Let V_1 and V_{-1} be the subspace of V consisting of all the eigenvectors corresponding to the eigenvalue $\lambda = 1$ and $\lambda = -1$, respectively. Prove that $V_1 \neq \{0\}$, $V_{-1} \neq \{0\}$, $V_{-1} \cap V_1 = \{0\}$, and $V_{-1} + V_1 = V$.
- Prove that there exists a basis for V such that the matrix representation F for f takes the form of a 2×2 block matrix

$$F = \begin{pmatrix} I_p & 0_{p \times q} \\ 0_{q \times p} & -I_q \end{pmatrix},$$

where I_p , I_q , $0_{p \times q}$, and $0_{q \times p}$, denote the $p \times p$ unit matrix, the $q \times q$ unit matrix, the $p \times q$ zero matrix, and the $q \times p$ zero matrix, respectively, and where $p + q = n$, with $p > 0$ and $q > 0$.

2. Let $V \neq \{0\}$ be a vector space over the real numbers \mathbb{R} with the dimension $\dim(V) = n < \infty$. Let $f : V \rightarrow V$ be a linear mapping. Assume that f has the property $f \circ f = -\text{id}_V$, where id_V denotes the identity mapping of V .

- Prove that f has no (real) eigenvalues.
- Let $e \in V$, $e \neq 0$. Prove that e and $f(e)$ are linearly independent.
- Prove that the dimension n of V must be even.

3. Consider the following matrices in $M_3(\mathbb{C})$.

$$(i) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad (ii) \begin{pmatrix} 3 & 0 & 0 \\ 2 & 3 & -1 \\ 0 & 0 & 3 \end{pmatrix} \quad (iii) \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$$

- Find the characteristic polynomials $C(x)$ for each of the matrices above.
- Find the minimal polynomials $M(x)$ for each of the matrices above.
- Find the Jordan canonical form for each of the matrices above.

4. a) Describe the automorphism group $\text{Aut}(\mathbb{Z}/9\mathbb{Z})$ of the group $\mathbb{Z}/9\mathbb{Z}$.

- Give an example of a group with the trivial center.
- Prove that if a group G has the trivial center then $|\text{Aut}(G)| \geq |G|$.

5. Let S_n denote the symmetric group on n letters.

- Find an element $a \in S_n$ of order n .
- Find an element $b \in S_5$ of order 6.
- Describe the conjugacy classes in S_4 . How many are there?

- 6.** Let G be the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by elements $(2, 6), (8, 21), (-4, -9)$. Find two generators for G .
- 7.** a) Let R be a commutative ring, $I \subset R$ an ideal such that R/I is a field. Let $a, b \in R$ such that $ab \in I$. Prove that $a \in I$ or $b \in I$.
b) Describe all maximal ideals in the polynomial ring $\mathbb{C}[x]$.
c) Show that the principal ideal $(y) \subset \mathbb{C}[x, y]$ is prime, but not maximal.
d) Show that the principal ideal $(y - x^2) \subset \mathbb{C}[x, y]$ is prime.
- 8.** Consider the ring $M_n(\mathbb{R})$ of $n \times n$ matrices with real entries.
a) Show that $M_n(\mathbb{R})$ is a simple ring (has no nontrivial two-sided ideals).
b) Let M be a left module over $M_n(\mathbb{R})$. Notice that the field \mathbb{R} is embedded in the ring $M_n(\mathbb{R})$ as scalar matrices; hence in particular M is an \mathbb{R} -vector space. Prove that $\dim_{\mathbb{R}}(M) \geq n$. (Hint: use a)).
- 9.** a) Let \mathbb{F}_q denote a finite field with q elements. Prove that $q = p^n$ for a prime p and some $n > 0$. (You cannot use the classification of finite fields.)
b) Describe the Galois group $Gal(\mathbb{F}_8/\mathbb{F}_2)$ explicitly.
c) Let $f(x) \in \mathbb{F}_2[x]$ be a polynomial of degree 5. Show that the field \mathbb{F}_{32} contains all the roots of $f(x)$.

Tier 1 Algebra Exam—August 2004

(5 pts) 1. Let A be a 5×5 matrix with entries in \mathbb{R} . If $A^3 = 0$ and

$$\dim(\ker(A)) + \dim(\ker(A^2)) = 8,$$

find the Jordan canonical form of A .

(5 pts) 2. Let e_1, e_2 be two (non-zero) eigenvectors of a linear transformation $T : V \rightarrow V$, and assume $e_1 \neq -e_2$. Show that $e_1 + e_2$ is an eigenvector of T if and only if e_1, e_2 correspond to the same eigenvalue.

(6 pts) 3. Given that the characteristic polynomial of a 5×5 integer matrix is $x(x^4 + 1)$, is the matrix diagonalizable over \mathbb{R} ? over \mathbb{C} ? In each case, give its diagonal form if it exists or explain why it is not diagonalizable.

(5 pts) 4. List all the abelian subgroups of S_4 , the permutation group on 4 elements.

(5 pts) 5. Is the group of automorphisms of an abelian group always abelian? Prove, or give a counterexample.

6. Let \mathbb{C}^* denote the group whose elements are $\mathbb{C} \setminus \{0\}$, equipped with multiplication.

(4 pts) (a) Show that for any group G and any abelian group H , the group operation of H induces an operation on $\text{Hom}(G, H)$ which makes it into an abelian group.

(7 pts) (b) Use the structure theorem for finite abelian groups to show that for any finite abelian group G , $\text{Hom}(G, \mathbb{C}^*)$ is isomorphic to G .

(6 pts) 7. Let H, K be proper subgroups of a group G , so that neither of them is contained in the other one. In each of the following two cases, prove that $H \cap K$ must be normal in G or prove that it cannot be normal in G or give examples showing that it can either be normal or not normal. (i) H, K both normal in G . (ii) H is normal in G , but K is not.

8. Let R be an integral domain. Prove:

(5 pts) (a) An element $a \in R$ is irreducible (meaning that a is not a unit, and whenever $a = bc$ for $b, c \in R$, either b or c is a unit) if and only if Ra is maximal among the proper principal ideals of R .

(5 pts) (b) If R is a principal ideal domain, then each irreducible element in R is prime.

(5 pts) 9. Show that $\mathbb{Z}[\sqrt{-7}]$ is not a UFD.

10. Let $R = F[x]$, where F is a field. Let $f(x)$ and $g(x)$ be polynomials in R such that the degree of $f(x)$ is smaller than the degree of $g(x)$. Let $g(x) = g_1(x) \cdot g_2(x)$ for relatively prime polynomials $g_i(x)$.

- (7 pts) (a) Show that there are polynomials $f_i(x)$ such that the degree of each $f_i(x)$ is smaller than the degree of $g_i(x)$ for $i = 1, 2$ and such that

$$\frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)} .$$

- (4 pts) (b) Are the $f_i(x)$ unique? Justify your answer.

- (4 pts) 11. Let p be a prime and let α be algebraic over \mathbb{Z}_p , the field with p elements. It is given that the multiplicative order of α is k .

Show that p does not divide k .

- (6 pts) 12. Find the irreducible polynomial of $\alpha = \sqrt{2} + \sqrt[3]{5}$ over \mathbb{Q} . Justify why the polynomial you give is α 's irreducible polynomial.

13. Let p be a prime. For a natural number r , let $GF(p^r)$ denote the finite field with p^r elements.

- (4 pts) (a) Prove one of the two implications in the following proposition:

Proposition. $GF(p^r) \subseteq GF(p^s)$ if and only if r divides s .

- (7 pts) (b) Assuming the proposition above, prove that there are $p^{15} - p^3 - p^5 + p$ elements α in the algebraic closure of \mathbb{Z}_p for which $\mathbb{Z}_p(\alpha)$ is $GF(p^{15})$.

14. Let K be an extension field of the field F , and let α be an element of K .

- (5 pts) (a) If α is transcendental, show that $F(\alpha) \neq F(\alpha^2)$. Give an example to show that the reverse implication is not true.

- (5 pts) (b) If α is algebraic over F and $F(\alpha) \neq F(\alpha^2)$, show that $[F(\alpha) : F]$ is even. Give an example to show that the reverse implication is not true.

ALGEBRA TIER ONE EXAMINATION, JANUARY 2005

- (1) (a) Compute the units in the ring $\mathbb{Z}_4[x]$.
(b) Find an irreducible polynomial of degree 4 in $\mathbb{Z}_2[x]$. Justify your answer.
- (2) Give examples of two 6×6 matrices A and B over \mathbb{Q} with minimal polynomial $(x - 2)^2(x^2 + 3)$ such that A is not similar to B .
- (3) Find an explicit formula for the entries of the following matrix in terms of n .

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$$

- (4) Let G be a group containing a normal subgroup H isomorphic to D_8 , the dihedral group of order 8. Prove that G must have a nontrivial center.
- (5) Let ϕ be a group homomorphism from D_{18} , the dihedral group of order 18, to S_8 . Prove that ϕ is not injective.
- (6) Let R be a unique factorization domain.
(a) Prove that if P is a nonzero prime ideal of R then P must contain an irreducible element.
(b) A prime ideal P in an integral domain R is called minimal if $P \neq 0$ and the only prime ideals Q such that $Q \subseteq P$ are $Q = 0$ and $Q = P$. Prove that in a unique factorization domain every minimal prime is principal.
- (7) Let F be a field and let $f(x) \in F[x]$ be a nonzero polynomial. Let n be the degree of $f(x)$. The quotient ring $F[x]/(f(x))$ may be viewed as an F -vector space via $\alpha(g(x) + (f(x))) = \alpha g(x) + (f(x))$ for all $\alpha \in F$ and $g(x) \in F[x]$. Prove that this vector space is finite dimensional over F and that the images of the elements $1, x, x^2, \dots, x^{n-1}$ form a basis.

- (8) Let R be a ring and let $N(R) = \{r \in R \mid r^k = 0 \text{ for some } k > 0\}$.
- (a) Prove that if R is commutative, then $N(R)$ is an ideal in R .
 - (b) Give an example to show that there are noncommutative rings R in which $N(R)$ is not an ideal.
- (9) Prove that for any prime p , the field \mathbb{F}_p with p elements contains an element a such that $[\mathbb{F}_p(\sqrt[3]{a}) : \mathbb{F}_p] = 3$ if and only if $p - 1$ is divisible by 3.
- (10) For each of the following construct an example or prove that there is none.
- (a) A finite field extension L/K and elements $\alpha, \beta \in L$ of degree 2 over K such that $\alpha + \beta$ is of degree 3 over K .
 - (b) A finite field extension L/K and elements $\gamma, \delta \in L$ of degree 3 over K such that $\gamma + \delta$ has degree 6 over K .

Algebra Tier 1 Exam August 2005

Time: 4 hours Points: 100

The point value for each problem is indicated. For problems with multiple parts, each part carries an equal point value unless indicated otherwise. Please provide an explanation for your answer to each problem, unless indicated otherwise.

1. (8 points) Given finite groups G_1, G_2 and H , along with surjective homomorphisms $\phi_i : G_i \rightarrow H$, define

$$P = \{(g_1, g_2) \in G_1 \times G_2 \mid \phi_1(g_1) = \phi_2(g_2)\}.$$

- a. Show that P is a subgroup and that the homomorphism $f: P \rightarrow G_1$ given by $f(g_1, g_2) = g_1$ is surjective.
- b. Find a relation between the orders of G_1, G_2, H , and P .
2. (8 points)
- a. What is the largest order of a cyclic subgroup of the symmetric group S_7 ?
- b. How many cyclic subgroups are there of the order found in part (a)?

3. (10 points) Let \mathbb{Z}_n denote the additive cyclic group of order n . Let G be the group $\mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{11}$ and let H be the subgroup $\{6g \mid g \in G\}$.

- a. Express the groups H and G/H as direct sums of cyclic groups.
- b. Is G isomorphic to $H \oplus G/H$?

4. (10 points) Let G be a group and H a subgroup of G . Suppose further that $g^2 \in H$ for all $g \in G$.

- a. Show that H is a normal subgroup of G .
- b. Give an example of a group G with a non-normal subgroup K such that $g^4 \in K$ for all $g \in G$.

5. (6 points) Find polynomials $u(x)$ and $v(x)$ in $\mathbb{Q}[x]$ such that

$$u(x)(x^3 - 5) + v(x)(x^2 + 2x + 3) = 1.$$

6. (6 points) Let R be any ring with 1 such that $r^2 = r$ for all $r \in R$. Show that the characteristic of R is 2 and that R is commutative.

7. Let D be an integral domain that is not a field.
- (2 points) Show that D has a nonzero proper ideal.
 - (6 points) Show further that D has infinitely many distinct ideals.
8. (6 points) Consider two subspaces W_1, W_2 of \mathbb{R}^3 , each spanned by 2 vectors: W_1 by $\{(1, 2, 1), (-1, -1, 0)\}$ and W_2 by $\{(2, 3, 3), (1, 1, -2)\}$. Find a basis for $W_1 \cap W_2$.
9. (6 points) Show that there does not exist a 4×4 matrix A with real entries which satisfies all of the following properties: (i) the characteristic polynomial is $(x + 2)^4$, (ii) $(A + 2I)^2 \neq 0$, and (iii) $\text{rank}(A + 2I)^2 = \text{rank}(A + 2I)^3$.
10. (8 points) Let V be a vector space of dimension greater than 1 over the real numbers. Let T be a linear transformation of V to itself of rank 1. Show that the minimal polynomial of T is $x(x - a)$ for some real number a .
11. (6 points) Let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^5 + 1$. If $\alpha \neq -1$, show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.
12. a. (6 points) Construct a field of 27 elements and find the structure of its additive group.
- b. (4 points) State without proof the structure of the multiplicative group of nonzero elements in that field.
13. (8 points) For each of the following, construct an example. No explanation is required.
- An infinite field of characteristic $p \neq 0$.
 - A field K and a proper subfield E , both of which are algebraically closed.

ALGEBRA TIER I
January 2006

Unless stated otherwise, all your answers require justification. A correct answer without a correct proof earns little credit. All questions are worth the same number of points.

1. Find the eigenvalues of the complex matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2. Consider the complex matrices

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- a) Find characteristic polynomials of A and B .
b) Does there exist an invertible matrix P such that $PAP^{-1} = B$?

3. Consider the complex matrices

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- a) Find the ranks of C and D .
b) Does there exist an invertible matrix P such that $PCP^{-1} = D$?

4. Let A be a $n \times n$ complex matrix such that A^2 is the identity matrix. Prove that there exists an invertible $n \times n$ matrix Q such that the matrix QAQ^{-1} is diagonal.

5. Let A be an invertible matrix, and let E be an upper triangular matrix with zeroes on the diagonal, that is: the (i, j) 'th entry of E is 0 for all $j \leq i$. Assume that $AE = EA$. Show that the matrix $A + E$ is invertible.

6. Give an example or compute, without proof, each of the following:
- All the abelian groups, up to isomorphism, of order 144 (list them).
 - An infinite group all of whose elements have finite order.
 - Groups $H \triangleleft K \triangleleft G$ (H is normal in K and K is normal in G) such that H is not normal in G .
7. Let G be a group, and let H and K be normal subgroups with $H \cap K = \{e\}$. If $h \in H$ and $k \in K$ then show that $hk = kh$.
8. Prove that the center of the finite symmetric group S_n is $\{e\}$ for $n \geq 3$. (Recall that the center is the set of elements which commute with every element of the group.)
9. How many conjugacy classes are there in the symmetric group S_5 ?
10. Show that if G has trivial center, then G is isomorphic to a subgroup of the group of automorphisms of G . (An *automorphism* of G is an isomorphism $\phi : G \rightarrow G$. Note that the set of automorphisms forms a group under composition.)
11. What are the units in the ring of Gaussian integers $\mathbb{Z}[i]$?
12. a) Let S be a subset of the complex numbers \mathbb{C} . Show that the intersection of all the fields $F \subseteq \mathbb{C}$ which contain S is a field.
- b) Let F_1, F_2 be subfields of \mathbb{C} , and assume that $[F_1 : \mathbb{Q}] = [F_2 : \mathbb{Q}] = 2$ and $F_1 \neq F_2$. Let F_3 be the minimal subfield of \mathbb{C} which contains $F_1 \cup F_2$. Show that $[F_3 : \mathbb{Q}] = 4$.
13. a) Let F be a finite field of order p^a . Show that every non-zero element $x \in F$ satisfies $x^{p^a-1} = 1$.
- b) Explain why the polynomial $x^n - 1$ can have at most n roots in F .
- c) Show that the non-zero elements of F , endowed with F 's multiplication, form a cyclic group of order $p^a - 1$.

Tier 1 Examination - Algebra

August 23, 2006

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by \mathbf{R} and the set of rational numbers by \mathbf{Q} .

(20) 1. Find an example of each of the following (no proof necessary):

(a) An infinite integral domain in which there are exactly 4 units.

(b) Two nonisomorphic nonabelian groups of order 12.

(c) A unique factorization domain with exactly one irreducible element (up to multiplication by a unit).

(d) An element of order 3 in $GL_2(\mathbf{Q})$.

(10)2. Find the sum of the reciprocals of the eigenvalues of the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

(10)3. Let n be a positive integer. Let $V_0, V_1, \dots, V_{2n-1}$ be a sequence of finite dimensional vector spaces. For $i = 0, 1, \dots, 2n$, let $T_i : V_i \rightarrow V_{i+1}$ be linear transformations, where by convention, $V_{2n} = V_0$ and $T_{2n} = T_0$. Suppose that for $i = 0, 1, \dots, 2n - 1$, we have

$$\ker(T_{i+1}) = \operatorname{im}(T_i)$$

Prove that

$$\dim(V_0) + \dim(V_2) + \dim(V_4) + \cdots + \dim(V_{2n-2}) = \dim(V_1) + \dim(V_3) + \cdots + \dim(V_{2n-1}).$$

(10)4. Let R be a ring with unit (possibly non-commutative). An element α in R is called *left quasi-invertible* if $1 - \alpha$ is left invertible, that is, if there exists $b \in R$ such that $b(1 - \alpha) = 1$. A subset of R is called left quasi-invertible if all of its elements are left quasi-invertible.

(a) Show that if α is in every maximal left ideal, then α is left quasi-invertible.

(b) Show that if the left ideal generated by α is left quasi-invertible, then α is contained in every maximal left ideal.

(15)5. Let R be a commutative ring. If I and J are ideals in R we define the product ideal to be $IJ = \{\sum_{k=1}^n x_k y_k \mid n \geq 1 \text{ and } x_k \in I, y_k \in J\}$ and we define the sum ideal to be $I + J = \{x + y \mid x \in I, y \in J\}$.

(a) Prove that IJ is an ideal in R .

(b) Prove that $IJ \subset I \cap J$ and give an example to show that equality does not always hold.

(c) Prove that if $I + J = R$ then $IJ = I \cap J$.

(10)6. (a) Let R be an integral domain containing a subring F such that F is a field and such that R is finite dimensional as a vector space over F . Show that R is a field.

(b) Let T be a field extension of the field F and let K and L be intermediate fields such that K and L are both finite dimensional over F . Let $KL = \{\sum_{k=1}^n x_k y_k \mid n \geq 1 \text{ and } x_k \in K, y_k \in L\}$. Prove KL is a subfield of T .

(10)7. Let H and K be subgroups of the group G . Prove that $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

(10)8. Let G be a group and x, y elements of order 2. Let H be the subgroup generated by x and y . Prove that the subgroup generated by xy is normal in H and has index two in H .

(10)9. Let F be a field, let $f(X)$ be a polynomial with coefficients in F , and let $R = F[X]/(f(X))$.

(a) Suppose F is the rational numbers and $f(X) = X^2 - 1$. Let α be the image of $a_0 + a_1X + \cdots + a_nX^n$ in R (for $a_0, \dots, a_n \in F$). Find concise necessary and sufficient conditions on a_0, \dots, a_n for α to be a unit.

(b) Let $f(X) = X^3 - 3X^2 - 1$. Show that if F is the real numbers, then R has zero divisors, but if F is the rational numbers, then R does not.

Tier 1 Algebra Examination, January, 2007

Important:

- Justify fully each answer unless otherwise directed.
- Notation: $\{1, 2, 3, \dots\} = \mathbb{N}$, \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} denote respectively the natural numbers, integers, rationals, reals, and complex numbers

1. (5 each pts.)

- (a) Give an example of groups G, H such that $\text{Aut}(G)$ and $\text{Aut}(H)$ are finite but $\text{Aut}(G \times H)$ is infinite. (Here, $\text{Aut}(G)$ denotes the set of automorphisms of G .) *No proof required.*
- (b) Give an example of an ideal in $\mathbb{Z}[x]$ that is prime but not maximal. *No proof required.*
- (c) Give an example of an integral domain that is not a unique factorization domain. *No proof required.*
- (d) State Eisenstein's criterion for a polynomial $f \in \mathbb{Z}[x]$ to be irreducible over \mathbb{Q} . *No proof required.*

2. (10 pts.) Let V be the real vector space of functions on \mathbb{R} spanned by the set of real-valued functions $\{e^x, xe^x, x^2e^x, e^{2x}\}$. Let $T : V \rightarrow V$ be the linear operator on V defined by $T(f) = f'$. Find (i) a Jordan canonical form of T , and (ii) a Jordan canonical basis.

3. (10 pts.) Let $f : V \rightarrow V$ be an endomorphism of a finite-dimensional vector space V . Show that there is a subspace U of V such that $f(U) = f(V)$ and $V = U \oplus \ker f$.

4. (10 pts.) Let G be a finitely generated abelian group. Prove that there are no nontrivial homomorphisms $\phi : \mathbb{Q} \rightarrow G$, where \mathbb{Q} denotes the additive rationals.

5. **(10 pts.)** Let G be a simple group of order n . Let H be a subgroup of G of index k with $H \neq G$. Show that n divides $k!$.
6. **(10 pts.)** Let R be a commutative ring with unity 1. Suppose each subring of R contains 1. Prove that R is a field of nonzero characteristic.
7. **(10 pts.)** Let R be a ring with 1, let $a \in R$, and suppose $a^n = 0$ for some $n \in \mathbb{N}$. Prove that $1 + a$ is a unit of R .
8. **(10 pts.)** Let R be a commutative ring with 1. Let m be a maximal ideal of R such that $m \cdot m = 0$.
- (a) Prove that m is the only maximal ideal of R .
- (b) Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ be a polynomial in $R[x]$ such that $a_i \in m$ for all i and $a_0 \neq 0$. Prove that $f(x)$ is irreducible, i.e., $f(x)$ is not a product of two polynomials in $R[x]$ of degree strictly smaller than $\deg f$.
9. **(10 pts.)** Let F denote a finite field of order $2^5 = 32$. Prove that for each integer $1 \leq n < 32$ and each $a \in F$, the equation $x^n = a$ has a solution in F .
10. **(10 pts.)** Determine with proof the degree of $\sqrt{2} + \sqrt{3}$ over \mathbb{Q} .

Tier I Algebra Exam
August, 2007

- **Be sure to fully justify all answers.**
 - **Notation** The sets of integers, rational numbers, real numbers, and complex numbers are denoted \mathbf{Z} , \mathbf{Q} , \mathbf{R} , and \mathbf{C} , respectively. All rings are understood to have a unit.
 - **Scoring** Each single part problem is worth 10 points. Each part of a multiple part problem is worth 5 points. (eg. Problem 1 is worth 10 points, Problem 2 is worth 25 points.)
 - **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
-

- (1) Prove that for a group G and positive integer k , if G contains an index k subgroup, then the intersection of all index k subgroups of G is a normal subgroup.
- (2) Let $F : G \rightarrow G$ be an endomorphism, that is, a homomorphism from the group G to itself. Let F^n denote the n -fold composition of F with itself, and let $K_n = \text{Kernel}(F^n)$.
- (a) Show that $K_n \subseteq K_{n+1}$ for all n .
- (b) Let $F : (\mathbf{Z}/16\mathbf{Z})^3 \rightarrow (\mathbf{Z}/16\mathbf{Z})^3$ be the endomorphism defined by $F(x, y, z) = (2z, 2x, 8y)$. For all $n \geq 1$, describe K_n as a direct sum of cyclic groups.
- (c) Show that if F is an endomorphism of the symmetric group S_5 , $K_{n+1} = K_n$ for all $n \geq 2$.
- (d) Give an example of an endomorphism F of the symmetric group S_5 for which $K_2 \neq K_1$.
- (e) Prove that for general G and F , if $K_n = K_{n+1}$, then $K_n = K_{n+i}$ for all $i \geq 0$.
- (3) Let $S = \{(x, y) \mid 23x + 31y = 1, x + y < 100\} \in \mathbf{Z}^2$. Find the element of S for which $x + y$ is as large as possible.
- (4) Let $T : \mathbf{R}^3 \rightarrow \mathbf{R}^4$ and $S : \mathbf{R}^4 \rightarrow \mathbf{R}^1$ be linear transformations given by:
- $$T(x, y, z) = (x + 2y + z, x - y + 4z, x - y + 4z, 2x + y + 5z)$$
- $$S(x, y, z, w) = (x - y + 2z - w).$$
- Find two sets of vectors in \mathbf{R}^4 , $\{\alpha_1, \dots, \alpha_m\}$ and $\{\beta_1, \dots, \beta_n\}$ such that $\{\alpha_1, \dots, \alpha_m\}$ is a basis of $\text{Im}(T)$ and $\{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n\}$ is a basis for $\text{Ker}(S)$. Justify your answer.
- (5) Let $T_1, T_2 : \mathbf{C}^3 \rightarrow \mathbf{C}^3$ be linear transformations. Show that if both linear transformations have minimal polynomials of degrees at most 2, then there is a vector that is an eigenvector for both T_1 and T_2 .
- (6) Let R be a commutative ring with unity and suppose that for every $r \in R$ there is an $n \geq 2$ so that $r^n = r$. Show that every prime ideal in R is maximal.
- (7) Suppose that R is an integral domain. Is it possible that R contains additive subgroups isomorphic to $\mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/q\mathbf{Z}$ for p and q distinct primes? Justify your answer.

- (8) Prove that the polynomial $2x^4 + x + 1 \in \mathbf{Q}[x]$ is irreducible. Justify all your work.
- (9) Let F_q denote the finite field with q elements. Show that for any $a \in F_q$ the equation $x^n = a$ has a solution in F_q if n is relatively prime to $q - 1$.
- (10) Let $p(t) = t^3 - 2 \in \mathbf{Q}[t]$. Let $\alpha = \sqrt[3]{2}$ be the real root of p and let β be a complex root of p . Determine if $\alpha \in \mathbf{Q}[\beta]$ and explain your answer.
- (11) Let $d > 1$ and let $p(x)$ and $q(x)$ be relatively prime irreducible polynomials in $\mathbf{Q}[x]$ of degree d . Suppose $p(\alpha) = 0 = q(\beta)$ for some $\alpha, \beta \in \mathbf{C}$. It follows that $1 \leq [\mathbf{Q}(\alpha, \beta) : \mathbf{Q}(\alpha)] \leq d$.
- (a) Find an example of a d, p, q, α , and β , so that $[\mathbf{Q}(\alpha, \beta) : \mathbf{Q}(\alpha)] = 1$.
- (b) Find an example of a d, p, q, α , and β , so that $[\mathbf{Q}(\alpha, \beta) : \mathbf{Q}(\alpha)] = d$.

Algebra Tier 1

January 2008

All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page.

Problem 1. Find eigenvalues and the corresponding eigenvectors of the complex matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 2. Let A be a 5×5 complex matrix such that $A^3 = 0$. List all possible Jordan canonical forms of A .

Problem 3. Find a 5×5 matrix A with rational entries whose minimal polynomial is $(x^3+1)(x+2)^2$.

Problem 4. Let A be a complex $n \times n$ matrix such that $A^m = I$ for some $m \geq 1$. Prove that A is conjugate to a diagonal matrix.

Problem 5. Consider the group $\mathbf{R}, +$, the additive group of the real numbers.

- Show that any homomorphism from a finite group to $\mathbf{R}, +$ has to be the trivial homomorphism.
- Show that any homomorphism from $\mathbf{R}, +$ to a finite group has to be the trivial homomorphism.

Problem 6. Consider the subgroup H of the group $\mathbf{Z}/12 \times \mathbf{Z}/12$ generated by the element (a^4, a^6) , where a is a generator of $\mathbf{Z}/12$.

- What is the order of H ? List its elements.
- How many elements are there in $(\mathbf{Z}/12 \times \mathbf{Z}/12)/H$?
- Write $(\mathbf{Z}/12 \times \mathbf{Z}/12)/H$ as a product of cyclic groups, each of which has order equal to a power of some prime. Find a generator for each of these cyclic subgroups.

Problem 7. Show that in a finite group of odd order every element is a square.

Problem 8. For each of the following subgroups of S_4 (the permutation group on four elements), say what its order is and justify your answer.

- The subgroup generated by $(1, 2)$ and $(3, 4)$.
- The subgroup generated by $(1, 2)$, $(3, 4)$, and $(1, 3)$.
- The subgroup generated by $(1, 2)$, $(3, 4)$, and $(1, 3)(2, 4)$.
- The subgroup generated by $(1, 2)$ and $(1, 3)$.

Problem 9. Let R be an integral domain that contains a field K . Show that if R is a finite dimensional vector space over K , then R is a field.

Problem 10. Let $f(x)$ be a polynomial with coefficients from a finite field F with q elements. Show that if $f(x)$ has no roots in F , then $f(x)$ and $x^q - x$ are relatively prime.

Problem 11. Let α be a root of an irreducible polynomial $x^3 - 2x + 2$ over \mathbf{Q} . Find the multiplicative inverse of $\alpha^2 + \alpha + 1$ in $\mathbf{Q}[\alpha]$ in the form $a + b\alpha + c\alpha^2$ with $a, b, c \in \mathbf{Q}$.

Problem 12. Let $f(x)$ and $g(x)$ be irreducible polynomials over $\mathbf{Q}[x]$. Let α be a root of $f(x)$ and let β be a root of $g(x)$. Show that $f(x)$ is irreducible over $\mathbf{Q}(\beta)$ if and only if $g(x)$ is irreducible over $\mathbf{Q}(\alpha)$.

TIER ONE EXAMINATION - ALGEBRA
AUGUST, 2008

Justify your answers. All rings are assumed to have an identity.
The numbers in parentheses are the points for that problem.

(9)1. Complete the following definitions:

- (a) Let G be a group and let $g \in G$. The order of g is
- (b) Let K/F be a field extension. An element $a \in K$ is called algebraic if
- (c) An ideal I in a commutative ring R is called prime if

(12)2. Let G be a group and let $G_2 = \{g^2 \mid g \in G\}$. Let H denote the intersection of all subgroups of G containing G_2 .

- (a) Prove that H is a normal subgroup of G .
- (b) Prove that G/H is abelian.
- (c) Prove that if G/H is finite, its order is a power of 2.

(10)3. Let K be a field. Let $a, b \in K$ and let $R = K[x]/(x^2 + ax + b)$. Prove that exactly one of the following is true:

- R is a field.
- R is isomorphic to K^2 , the direct sum of two copies of K .
- There is a nonzero element $r \in R$ such that $r^2 = 0$.

(8)4. A complex matrix A has characteristic polynomial $(x - 2)^4(x + 2)$ and minimal polynomial $(x - 2)(x + 2)$. Determine the possible Jordan canonical forms for A .

(12)5. Let V be an n -dimensional real vector space.

(a) Let a, b be nonnegative integers. Prove there are subspaces V_a and V_b of dimension a, b respectively with $V_a \cap V_b = 0$ if and only if $a + b \leq n$.

(b) Let a, b, c be nonnegative integers. Prove there are subspaces V_a, V_b and V_c of dimension a, b, c respectively with $V_a \cap V_b \cap V_c = 0$ if and only if $a \leq n, b \leq n, c \leq n$ and $a + b + c \leq 2n$.

(8)6. Let F be a field. Determine the possible finite groups G that are isomorphic to a subgroup of F^+ , the additive group of F .

(10)7. A nonzero prime ideal P in a commutative ring R is called minimal if the only nonzero prime ideal Q contained in P is P itself. Now let F be a field and let $R = F[x, y]$, the polynomial ring in two variables over F . Prove that if P is a minimal prime ideal of R there is an irreducible element $f(x, y)$ in R such that $P = (f(x, y))$.

(10)8. Let D_n denote the dihedral group of order $2n$ (that is, D_n is the group of symmetries of the regular n -gon). Let G be a finite group. Prove that if there is a nontrivial homomorphism from D_n to G then the order of G is even.

(10)9. Let G be a group and let M, N be normal subgroups such that $MN = G$ and $M \cap N = \{e\}$. Prove that G is isomorphic to the direct product $G/M \times G/N$.

(10)10. Let $M_n(\mathbf{Q})$ denote the ring of $n \times n$ matrices over the rationals. Let K be a subring of $M_n(\mathbf{Q})$ such that K is a field and K contains \mathbf{Q} . Prove that the degree $[K : \mathbf{Q}]$ is finite and $[K : \mathbf{Q}]$ divides n .

ALGEBRA TIER 1 EXAM
JANUARY, 2009

Instructions: Each problem or major part of a problem counts five points, as indicated, for a total of 90 points. Start each problem on a new sheet of paper. Unless otherwise stated you should show your work and justify your claims.

Problem 1 (10 points).

- (1) Prove that every subgroup of a cyclic group is cyclic.
- (2) Determine, up to isomorphism, the finitely generated abelian groups with the property that every proper subgroup is cyclic.

Problem 2 (5 points). Let G be a group. Define a subgroup $H \leq G$ to be characteristic if for every isomorphism $\varphi : G \rightarrow G$ one has $\varphi(H) \subseteq H$. Now suppose that $H \leq G$ is a normal subgroup and $K \leq H$ is a characteristic subgroup of H . Prove that K is a normal subgroup of G .

Problem 3 (5 points). Let G be a finite abelian group of order n , in which the group operation is written multiplicatively. Suppose that the map $f : x \mapsto x^m$ is an automorphism of G , for some positive integer m . Prove that $\gcd(n, m) = 1$.

Problem 4 (10 points). Let V denote a finite-dimensional complex vector space, and $L(V)$ the complex vector space of linear transformations from V to itself. Given $A \in L(V)$, the function $T_A : L(V) \rightarrow L(V)$ defined by $T_A(X) = AX - XA$ for all $X \in L(V)$ is a linear transformation.

- (1) Suppose that $A, B \in L(V)$ have the same Jordan canonical form. Prove that T_A and T_B have the same Jordan canonical form.
- (2) Suppose that the dimension of V is equal to 2. Prove that for every $A \in L(V)$, the rank of the transformation T_A is either 0 or 2.

Problem 5 (5 points). Let V be a real vector space and $T : V \rightarrow V$ a linear transformation. Suppose every nonzero vector in V is an eigenvector of T . Prove that T is a scalar multiple of the identity.

Problem 6 (10 points).

- (1) Let A be a 2×2 matrix with entries in the field of real numbers such that $A^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Show that A is similar over the real numbers to the matrix $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
- (2) Let A be an $n \times n$ matrix with entries in the field of real numbers such that $A^2 + I_n = 0$ where I_n is the $n \times n$ identity matrix. Show that n is even and A is similar over the real numbers to the matrix $B = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$, where $n = 2m$.

Problem 7 (10 points). Let R be a commutative ring with 1 that contains exactly three ideals.

- (1) Show that every nonzero element of R is either a unit or a zero divisor.
- (2) Is the converse true? Justify your answer.

Problem 8 (5 points). For a prime integer p , let F_p denote the field of p elements. Suppose the greatest common divisor of the polynomials $f(x) = 6x^3 + 10x^2 - 10x + 16$ and $g(x) = 6x^2 + 10x - 16$ in $F_p[x]$ is 1. Find p .

Problem 9 (10 points). Let F_3 denote the finite field with 3 elements and \overline{F}_3 be its algebraic closure. Let K be the splitting field of $g(x) = x^{21} - 1$.

- (1) Find the number of zeros of $g(x)$ in the field \overline{F}_3 .
- (2) (a) Find the number of elements in K . (b) What are the numbers of elements in the maximal proper subfields of K ? (A subfield of K is said to be proper if it is not equal to K itself.)

Problem 10 (10 points).

- (1) Suppose γ is a complex number such that γ^2 is algebraic over \mathbb{Q} . Show that γ is algebraic over \mathbb{Q} .
- (2) Let α, β be complex numbers such that α is transcendental over \mathbb{Q} . Show that at least one of $\alpha - \beta$ and $\alpha\beta$ is transcendental.

Problem 11 (10 points). Let D be a domain. Two non-zero ideals $I, J \subset D$ are called “comaximal” if $I + J = D$ and the two ideals are called “coprime” if $I \cap J = I \cdot J$.

- (1) Show that if the ideals $I, J \subset D$ are comaximal, then they are coprime.
- (2) Show that if D is a principal ideal domain and the ideals $I, J \subset D$ are coprime, then they are comaximal.

Tier I Algebra Exam
August, 2009

- **Be sure to fully justify all answers.**
 - **Notation** The sets of integers, rational numbers, and real numbers are denoted \mathbf{Z} , \mathbf{Q} , and \mathbf{R} , respectively. For a prime integer p , \mathbf{Z}/p denotes the quotient $\mathbf{Z}/p\mathbf{Z}$. All rings are understood to have a unit.
 - **Scoring** Each problem is worth 10 points.
 - **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
-

- (1) Let A and B be finite subgroups of a group G of relatively prime orders; that is, $\gcd(|A|, |B|) = 1$. Prove that the function $\phi : A \times B \rightarrow G$ defined by $\phi(a, b) = ab$ is injective.
- (2) Find an element of largest order in the symmetric group S_{12} . Justify your answer.
- (3) (a) Let G and H be abelian groups and let $\text{Hom}(G, H)$ denote the set of all group homomorphisms from G to H . If $\phi, \psi \in \text{Hom}(G, H)$, define $\phi + \psi$ by $(\phi + \psi)(g) = \phi(g) + \psi(g)$ for all $g \in G$. Prove that $\text{Hom}(G, H)$ is an abelian group.
(b) Let C be a cyclic group such that $\text{Hom}(C, \mathbf{Z}/p) \cong \mathbf{Z}/p$. What can you say about the order of C ?
(c) Let G be a finitely generated abelian group and $\text{Hom}(G, \mathbf{Z}/p) \cong (\mathbf{Z}/p)^n$. What does this tell you about G ?
- (4) Let A, B be commuting 2×2 real matrices with characteristic polynomials $x^2 - 3x + 2$ and $x^2 - 1$, respectively. Show that either $A + B$ or $A - B$ has determinant 0.
- (5) Suppose that T is a linear transformation of a finite dimensional real vector space V having characteristic polynomial $f(t)g(t)$ where f and g are relatively prime. Show that $V = \text{Ker}(f(T)) \oplus \text{Ker}(g(T))$.
- (6) Let T be a linear transformation of a finite dimensional real vector space V and assume that V is spanned by eigenvectors of T . If $T(W) \subset W$ for some subspace $W \subset V$, show that W is spanned by eigenvectors. (Hint: consider the minimal polynomial of T .)

- (7) Let F denote the field with two elements and let E be an extension field of F .
- Show that if $\alpha \in E$ satisfies $f(\alpha) = 0$ for some $f \in F[x]$, then $f(\alpha^2) = 0$.
 - Suppose that $\alpha \in E$ is a root of the polynomial $f(x) = x^5 + x^2 + 1 \in F[x]$. List all roots of $f(x)$ in E in the form $a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4$ with each $a_i = 0$ or 1 .
- (8) Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be irreducible rational polynomials, having roots α and β , respectively. Find necessary and sufficient conditions on the coefficients (a, b, c, d) that imply that $\mathbf{Q}(\alpha)$ is isomorphic to $\mathbf{Q}(\beta)$. Prove that your conditions are both necessary and sufficient.
- (9) (a) Is the ring $\mathbf{Z}[i]$ of Gaussian integers an integral domain? Justify your answer.
- (b) Let $R = \mathbf{Z}[T]/\langle T^4 - 1 \rangle$, where $\langle T^4 - 1 \rangle$ is the ideal generated by $T^4 - 1$. Is R an integral domain? Justify your answer.
- (c) Show that sending T to i determines a ring homomorphism $\psi : R \rightarrow \mathbf{Z}[i]$. Describe $\text{Ker } \psi$ as all elements $a + bT + cT^2 + dT^3 \in R$ where a, b, c, d satisfy some conditions.
- (10) Let F^* denote the multiplicative group of all nonzero elements of a finite field F . Show that F^* is cyclic.

Algebra Tier 1

January 2010

All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page. The notation \mathbf{Z} , \mathbf{Q} , \mathbf{C} stands for integers, rational numbers and complex numbers respectively.

Problem 1. Let A be a $n \times n$ complex matrix which does not have eigenvalue -1 . Show that the matrix $A + I_n$ is invertible. (I_n is the identity $n \times n$ matrix.)

Problem 2. (a) Find the eigenvalues of the complex matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

(b) Find the eigenvectors of A .

(c) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Problem 3. Let A, B be $n \times n$ complex matrices such that $AB = BA$. Prove that there exists a vector $v \neq 0$ in \mathbf{C}^n which is an eigenvector for A and for B .

Problem 4. Suppose that G is a group of order 60 that has 5 conjugacy classes of orders 1, 15, 20, 12, 12. Prove that G is a simple group.

Problem 5. Prove that any group of order 49 is abelian.

Problem 6. How many conjugacy classes are there in the symmetric group S_5 ?

Problem 7. Let $G = GL_2(\mathbf{F}_5)$, the group of invertible 2×2 matrices with entries in the field \mathbf{F}_5 with 5 elements. What is the order of G ?

Problem 8. Let G and H be any pair of groups and let $S = \text{Hom}(G, H)$ denote the set of homomorphisms from G to H .

a) Prove that if H is an abelian group, then the operation “+” on S given by $(f_1 + f_2)(g) = f_1(g) + f_2(g)$ makes S into an abelian group.

b) Prove that if G is a finite cyclic group, then $\text{Hom}(G, \mathbf{Q}/\mathbf{Z})$ is isomorphic to G .

c) Find an infinite abelian group G so that $\text{Hom}(G, \mathbf{Q}/\mathbf{Z})$ is not isomorphic to G .

Problem 9. Describe the prime ideals in the ring $\mathbf{C}[x]$.

Problem 10. Find the degree of the minimal polynomial of $\alpha = \sqrt{2} + \sqrt[3]{3}$ over \mathbf{Q} .

Problem 11. a) Prove that the polynomial $x^2 + x + 1$ is irreducible over the field \mathbf{F}_2 with two elements.

b) Factor $x^9 - x$ into irreducible polynomials in $\mathbf{F}_3[x]$, where \mathbf{F}_3 is the field with three elements.

Problem 12. Determine the following ideals in \mathbf{Z} by giving generators:

$$(2) + (3), \quad (4) + (6), \quad (2) \cap (3), \quad (4) \cap (6)$$

Problem 13. Let $f(x) \in \mathbf{C}[x]$ be a polynomial of degree n such that f and f' (the derivative of f) have no common roots. Show that the quotient ring $\mathbf{C}[x]/(f)$ is isomorphic to $\mathbf{C} \times \dots \times \mathbf{C}$ (n times).

Algebra Tier I Exam
August 2010

1. Find all irreducible monic quadratic polynomials in $\mathbb{Z}_3[x]$.
(*Monic: coefficient of the highest power of x is one.*)
2. Let G be a finite group and $\Phi : G \rightarrow G$ an automorphism.
 - (a) Show that Φ maps a conjugacy class of G into a conjugacy class of G .
 - (b) Give a concrete example of non-trivial G and Φ such that $\{e\}$ is the only conjugacy class of G that Φ maps into itself. Explain.
 - (c) Show that if $G = S_5$ (the symmetric group on five letters), then g and $\Phi(g)$ must be conjugate for any $g \in G$.
3. Let V and W be real vector spaces, and let $T : V \rightarrow W$ be a linear map. If the dimensions of V and W are 3 and 5, respectively, then for any bases B of V and B' of W , we can represent T by a 5×3 matrix $A_{T,B,B'}$. Find a set S of 5×3 matrices *as small as possible* such that for any $T : V \rightarrow W$ there are bases B of V and B' of W such that $A_{T,B,B'} \in S$.
4. Is it possible to find a field F with at most 100 elements so that F has exactly five different proper subfields? If so, find all such fields. If not, prove that no such field F exists.
5. Let G be the group of rigid motions (more specifically, rotations) in \mathbb{R}^3 generated by $x =$ a 90° degree rotation about the x -axis, and $y =$ a 90° degree rotation about the y -axis.
 - (a) How many elements does G have?
 - (b) Show that the subgroup generated by x^2 and y^2 is a normal subgroup of G .
6. In this problem, R is a finite commutative ring with identity. Define $a \in R$ to be *periodic of period k* if a, a^2, \dots, a^k are all different, but $a^{k+1} = a$.
 - (a) In $R = \mathbb{Z}_{76}$, find an element $a \neq 0, 1$ of period 1.

- (b) In the same ring $R = \mathbb{Z}_{76}$ find an element that is not periodic.
 - (c) In $R = \mathbb{Z}_{76}$, list the possible periods and the number of elements of each period.
7. In this problem, R is a finite commutative ring with identity. Let $p(x) \in R[x]$, the ring of polynomials over R .
- (a) Show that $a \in R$ is a root of $p(x)$ if and only if $p(x)$ can be written as $p(x) = (x - a)g(x)$ with $g(x) \in R[x]$ of degree one less than the degree of $p(x)$.
 - (b) Prove or give a counterexample: A polynomial $p(x) \in R[x]$ of degree n can have at most n distinct roots in R .
8. Consider S_5 , the symmetric group on 5 letters. If $\sigma \in S_5$ has order 6, how many elements of S_5 commute with σ ?
9. Let A be a 5×5 *real* matrix of rank 2 having $\lambda = -i$ as one of its eigenvalues. Show that $A^3 = -A$ and that A is diagonalizable (as a complex matrix).
10. (a) Give an example of an irreducible monic polynomial of degree 4 in $\mathbb{Z}[x]$ that is reducible in the field $\mathbb{Q}[\sqrt{2}]$. Explain why your example has the stated properties.
- (b) Show that there is *no* irreducible monic polynomial of degree 5 in $\mathbb{Z}[x]$ that is reducible in the field $\mathbb{Q}[\sqrt{2}]$.
11. Let M be the ring of 3×3 matrices with integer entries. Find all maximal two-sided ideals of M .
12. For which values of n in \mathbb{Z} does the ring $\mathbb{Z}[x]/(x^3 + nx + 3)$ have *no* zero divisors?

Tier 1 Algebra Exam

January 2011

1. For an element g of a group G , define the *centralizer subgroup* $C(g) = \{h \in G \mid hg = gh\}$. What is the minimal order of the centralizer subgroup of an element of order 2 in S_6 ? Explain.
2. Let G be a group and H_3 and H_5 normal subgroups of G of index 3 and 5 respectively. Prove that every element $g \in G$ can be written in the form $g = h_3h_5$, with $h_3 \in H_3$ and $h_5 \in H_5$.
3. Show that every finite group whose order is at least 3 has a non-trivial automorphism.
4. The following matrix has four distinct real eigenvalues. Find their sum and their product.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

5. Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be the linear transformation $T(a, b, c, d) = (a + b - c, c + d)$. Find a basis for the null space.
6. A 5×5 matrix A satisfies the equation $(A - 2I)^3(A + 2I)^2 = 0$. Assume that there are at least two linearly independent vectors v that satisfy $Av = 2v$. What are the possibilities for the Jordan canonical form? List only one in each conjugacy class.
7. Prove that in a commutative ring with a finite number of elements, prime ideals are maximal.

8. Let \mathbb{F}_4 be the finite field with four elements. Express

$$\mathbb{F}_4[x]/\langle x^4 + x^3 + x^2 + 1 \rangle$$

as a product of fields. Prove your result.

9. Recall an element r of a ring R is a *unit* if there is an $s \in R$ so that $rs = 1 = sr$ and an element r of a ring R is *nilpotent* if there is a positive integer n so that $r^n = 0$.
- (a) Give an example of a ring R and a unit $r \in R$ with $r \neq 1$.
 - (b) Give an example of a ring R and a nilpotent element $r \in R$ with $r \neq 0$.
 - (c) Show that for any ring R and for any element $r \in R$, that r is a nilpotent element of R if and only if $1 - rx$ is a unit in the polynomial ring $R[x]$.
10. Let $M_n(\mathbb{C})$ denote the vector space over \mathbb{C} of all $n \times n$ complex matrices. Prove that if M is a complex $n \times n$ matrix, then $C(M) = \{A \in M_n(\mathbb{C}) \mid AM = MA\}$ is a subspace of $M_n(\mathbb{C})$ of dimension at least n .

Tier 1 Algebra Exam
August 2011

Do all 12 problems.

1. (8 points) Let A be a matrix in $GL_n(\mathbb{C})$. Show that if A has finite order (i.e., A^k is the identity matrix for some $k \geq 1$), then A is diagonalizable.
2. (8 points) Let V be a finite-dimensional real vector space of dimension n . Define an equivalence relation \sim on the set $\text{End}_{\mathbb{R}}(V)$ of \mathbb{R} -linear homomorphisms $V \rightarrow V$ as follows: if $S, T \in \text{End}_{\mathbb{R}}(V)$ then $S \sim T$ if and only if there are invertible maps $A : V \rightarrow V$ and $B : V \rightarrow V$ such that $S = BTA$. (You may assume this is an equivalence relation.)
Determine, as a function of n , the number of equivalence classes.
3. (8 points) Let $n \geq 2$. Let A be the n -by- n matrix with zeros on the diagonal and ones everywhere else. Find the characteristic polynomial of A .

4. (8 points) Find the Jordan canonical form of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}$.

Justify your answer.

5. (8 points) Let $R = K[x, y, z]/(x^2 - yz)$, where K is a field. Show that R is an integral domain, but not a unique factorization domain.
6. (8 points) Let P be a prime ideal in a commutative ring R with 1, and let $f(x) \in R[x]$ be a polynomial of positive degree. Prove the following statement: if all but the leading coefficient of $f(x)$ are in P and $f(x) = g(x)h(x)$, for some non-constant polynomials $g(x), h(x) \in R[x]$, then the constant term of $f(x)$ is in P^2 .

[We recall that P^2 is the ideal generated by all elements of the form ab , where $a, b \in P$.h]

7. (10 points) Let p be a prime number and denote by $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field with p elements. For a positive integer n let \mathbb{F}_{p^n} be the splitting field of $x^{p^n} - x \in \mathbb{F}_p[x]$. Prove that the following statements are equivalent:
- 1) $k|n$.
 - 2) $(p^k - 1)|(p^n - 1)$.
 - 3) $\mathbb{F}_{p^k} \subset \mathbb{F}_{p^n}$.
8. (10 points) i) Show that $x^3 - 2$ and $x^5 - 2$ are irreducible over \mathbb{Q} .
 ii) How many field homomorphisms are there from $\mathbb{Q}[\sqrt[3]{2}, \sqrt[5]{2}]$ to \mathbb{C} ?
 iii) Prove that the degree of $\sqrt[3]{2} + \sqrt[5]{2}$ over \mathbb{Q} is 15.
9. (8 points) Let p be a prime number. Prove that any group of order p^2 is abelian.
10. (8 points) Let a be an element of a group G . Prove that a commutes with each of its conjugates in G if and only if a belongs to an abelian normal subgroup of G .
11. (8 points) Find the cardinality of $\text{Hom}(\mathbb{Z}/20\mathbb{Z}, \mathbb{Z}/50\mathbb{Z})$, where $\text{Hom}(\cdot, \cdot)$ denotes the set of group homomorphisms.
12. (8 points) Let G be a finite group, and let $M \subset G$ be a *maximal* subgroup, i.e., M is a proper subgroup of G and there is no subgroup M' such that $M \subsetneq M' \subsetneq G$. Show that if M is a normal subgroup of G then $|G : M|$ is prime.

[Hint. Consider the homomorphism $G \rightarrow G/M$.]

Algebra Tier 1

January 2012

All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page. The notation \mathbf{F}_n , \mathbf{R} , \mathbf{Z} stands for the field with n elements, the field of real numbers, and the ring of integers respectively.

Problem 1. Find the Jordan canonical form of the complex matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Problem 2. Find the matrix A^{2011} , where A is the matrix from problem 1 above.

Problem 3. Find the eigenvalues and a basis for the eigenspaces of the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 4. Find the matrix $e^C := I + C + \frac{C^2}{2!} + \frac{C^3}{3!} + \dots$, where

$$C = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

Problem 5. Let a and b be elements of a group G . Prove that ab and ba have the same order.

Problem 6. Prove that if G is a nonabelian group, then $G/Z(G)$ is not cyclic. Here $Z(G)$ denotes the center of G .

Problem 7. Prove that if G is a finite nonabelian group of order p^3 , where p is a prime, then $Z(G) = [G, G]$, where $Z(G)$ denotes the center of G and $[G, G]$ denotes the commutator subgroup of G .

Problem 8. Let C_2 denote a cyclic group of order 2. Determine the group $\text{Aut}(C_2 \times C_2)$, calculating its order and identifying it with a familiar group.

Problem 9. Find all irreducible polynomials of degree ≤ 4 in $\mathbf{F}_2[x]$.

Problem 10. Find the set of polynomials in $\mathbf{F}_2[x]$ which are the minimal polynomials of elements in \mathbf{F}_{16} .

Problem 11. Prove that the rings \mathbf{F}_{16} , $\mathbf{F}_4 \times \mathbf{F}_4$, and $\mathbf{Z}/16\mathbf{Z}$ are pairwise non-isomorphic.

Problem 12. Find all the maximal ideals in the ring $\mathbf{R}[x]$.

Problem 13. Let R be the ring of Gaussian integers and $I \subset R$ be an ideal. If R/I has 4 elements what are the possibilities for I and R/I ?

ALGEBRA TIER I

August, 2012

All answers must be justified. A correct answer without justification will receive little credit. Each problem is worth 10 points.

- (1) Let $f : \mathbb{Z}^3 \rightarrow \mathbb{Z}^4$ be the function

$$f(a, b, c) = (a + b + c, a + 3b + c, a + b + 5c, 4a + 8b)$$

- (a) Prove that f is a group homomorphism.
 (b) Let H denote the image of f . Find an element of infinite order in \mathbb{Z}^4/H .
 (c) Calculate the order of the torsion subgroup of \mathbb{Z}^4/H .

- (2) Let A be the matrix

$$A = \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

Calculate $[K : \mathbb{Q}]$ where K is the smallest subfield of \mathbb{C} containing \mathbb{Q} and the eigenvalues of A .

- (3) Let $R = \mathbb{Z}[x]/I$, where I is the ideal generated by $x^2 - 5x - 2$. Let S denote the ring of 2×2 integer matrices: $S = M_2(\mathbb{Z})$. Let B denote the matrix

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

- (a) Show that there exists a unique ring homomorphism $f : R \rightarrow S$ satisfying $f(x + I) = B$.
 (b) Let J denote the ideal in R generated by the element $(x - 1) + I$. Is $f(J)$ an ideal in S ? Why or why not?

- (4) Does there exist a non-abelian group of order 2012?

- (5) Let $n \in \{2, 3, 7\}$ and consider the ring

$$R_n = (\mathbb{Z}/n)[x]/(x^3 + x^2 + x + 2).$$

For which $n \in \{2, 3, 7\}$ (if any) is R_n a field? For which n (if any) is R_n a integral domain but not a field? For which n (if any) is R_n not an integral domain? Justify all your conclusions.

- (6) Let R be the subring of \mathbb{R} given by $R = \{n + m\sqrt{-10} \mid m, n \in \mathbb{Z}\}$. Show that the element $2 - \sqrt{-10}$ is irreducible in R but not prime.
- (7) Let G be a finite abelian group and let $\phi : G \rightarrow G$ be a group homomorphism. Note that for all positive integers k the function $\phi^k = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{k \text{ times}}$ is also a homomorphism from G to G . Prove there is a positive integer n such that $G \cong \ker(\phi^n) \times \phi^n(G)$.
- (8) Let G be a group containing normal subgroups of order 3 and 5. Prove G contains an element of order 15.
- (9) Let M be the following matrix:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ -4 & 1 & 0 \\ 3 & -2 & 3 \end{pmatrix}$$

Prove or disprove (by giving a counterexample) each of the following statements:

- (a) For every 3×4 complex matrix N there is a nonzero vector $v \in \mathbb{C}^4$ such that $MNv = 0$.
- (b) For every 3×4 complex matrix N there is a nonzero vector $v \in \mathbb{C}^3$ such that $NMv = 0$.
- (10) Suppose that K/F is a finite extension of fields and p is the smallest prime dividing $[K : F]$. Prove that for all $\alpha \in K$, $F(\alpha) = F(\alpha^{p-1})$.

Tier I Algebra Exam

January, 2013

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- **Be sure to fully justify all answers.**
 - **Notation** The sets of integers, rational numbers, real numbers, and complex numbers are denoted \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} , respectively. All rings are understood to have a unit and ring homomorphisms to be unit preserving.
 - **Scoring** Each problem is worth 10 points.
 - **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number and your test number on each sheet of paper.**
-

1. Give examples with *brief* justification:
 - (a) A commutative ring with exactly one non-zero prime ideal.
 - (b) A commutative ring with a non-zero prime ideal that is not maximal.
 - (c) A UFD that is not a PID.
 - (d) A 2×2 integer matrix having $1 + \sqrt{2}$ as an eigenvalue.
 - (e) A polynomial of degree 4 with integer coefficients that is irreducible over the rational numbers but not irreducible when reduced mod 3, mod 5, and mod 7.
2. Let R be a commutative ring with unit and let $P < R$ be a prime ideal. Show that if R/P is a finite set, then P is a maximal ideal.
3. Find the degree of the field $\mathbb{Q}(\sqrt[4]{2})$ as an extension of the field $\mathbb{Q}(\sqrt{2})$.
4. Let \mathbb{F} be a field with 8 elements and \mathbb{E} a field with 32 elements. Construct a (unit preserving) homomorphism of rings $\mathbb{F} \rightarrow \mathbb{E}$ or prove that one cannot exist.
5. In \mathbb{R}^5 , consider the subspaces
 $V = \langle (1, 2, 3, 3, 2), (0, 1, 0, 1, 1) \rangle$ and $W = \langle (0, -1, 3, 2, -1), (1, 1, 0, -1, 1) \rangle$,
where $\langle \rangle$ indicates span. Find a basis for $V \cap W$.
6. Compute $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}^{100}$.

Test continues on other side.

7. Consider the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For which $n \in \mathbb{Z}$ does there exist a matrix P (with entries in \mathbb{C}) such that $P^n = M$?

8. Suppose that ϕ is a homomorphism from $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ to itself satisfying $\phi^5 = \text{id}$ (where $\phi^5 = \phi \circ \phi \circ \phi \circ \phi \circ \phi$). Show that ϕ is the identity.
9. Consider the quotient of additive abelian groups $G = \mathbb{Q}/\mathbb{Z}$. Prove that every finite subgroup of G is cyclic.
10. Consider the order 2 subgroup $H = \{(1), (1\ 2)(3\ 4)\}$ of the symmetric group S_4 .
 - (a) What is the normalizer $N(H)$?
 - (b) What numbers occur as orders of non-identity elements of the quotient group $N(H)/H$?
11. Classify up to isomorphism all groups with 38 elements: Give a list of non-isomorphic groups with 38 elements such that every group with 38 elements is isomorphic to one in your list. Be sure to justify that your list consists of non-isomorphic groups and that you have identified all groups with 38 elements up to isomorphism.
12. For an abelian group A and a positive integer n , consider the automorphism of A given by multiplication by n . Denote by ${}_nA$ and A/n its kernel and cokernel (quotient), respectively. Let $\phi: A \rightarrow B$ be a homomorphism of finite abelian groups, and assume that for all prime numbers p , ϕ induces an isomorphism ${}_pA \rightarrow {}_pB$ and an isomorphism $A/p \rightarrow B/p$. Show that ϕ is an isomorphism.

Tier 1 Algebra Exam

August, 2013

- (8 points) Prove Fermat's Little Theorem: If p is a prime number and a is any integer, then $a^p - a$ is divisible by p .
- (8 points) Compute A^{2013} , where $A = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$.
- (8 points) Construct an explicit isomorphism between $\mathbb{Z}[\sqrt{-1}]/(7)$ and $\mathbb{Z}[\sqrt{-2}]/(7)$, where (7) denotes the ideal generated by 7.
- (9 points) Let $\mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order $n > 1$.
 - Show that the automorphism group $A = \text{Aut}(\mathbb{Z}/n\mathbb{Z})$ is abelian.
 - What is the order of the automorphism group of the finite group $G = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/25\mathbb{Z}$?
 - Let G be as in part (b). Is the group $\text{Aut}(G)$ abelian?
- (8 points) Show that $x^5 - (3 + i)x + 2$ is irreducible in $(\mathbb{Z}[i])[x]$.
- (8 points) For any pair of real numbers a and b , let $M_{a,b}$ be the $n \times n$ matrix

$$M_{a,b} = \begin{pmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a \end{pmatrix}$$

with entries a on the diagonal and b off the diagonal. Find the eigenvalues of $M_{a,b}$ and their multiplicities.

- (10 points) Classify (up to isomorphism) all groups of order 8. (You may use the following fact without proof: if $g^2 = 1$ for each element g in a group G , then G is abelian.)

8. (8 points) Let K be an algebraically closed field of characteristic $p > 0$, and let $q = p^n$. Show that the solutions of the equation $x^q = x$ form a subfield $F \subseteq K$.
9. (8 points) Let $M \in \mathcal{M}_n(\mathbb{C})$ be a diagonalizable complex $n \times n$ matrix such that M is similar to its complex conjugate \overline{M} ; i.e., there exists $g \in GL_n(\mathbb{C})$ such that $\overline{M} = gMg^{-1}$. Prove that M is similar to a real matrix $M_0 \in \mathcal{M}_n(\mathbb{R})$.
10. (8 points) Let $p > 2$ be an odd prime. Let F be a field with $q = p^n$ elements. How many solutions of the equation

$$x^2 - y^2 = 1$$

are there with $x, y \in F$?

11. (8 points) Let G be a group. Let t be the number of subgroups of G that are not normal. Prove that $t \neq 1$.
12. (9 points) Let V be a vector space of dimension n over a finite field F with q elements.
- Find the number of 1-dimensional subspaces of V .
 - Find the number of $n \times n$ invertible matrices with entries from F .
 - For each k , $1 \leq k \leq n$, find the number of k -dimensional subspaces of V .

Tier 1 Examination - Algebra

January, 2014

Justify all answers!-except in problem 1. All rings are assumed to have an identity element. The set of real numbers is denoted by \mathbf{R} , the complexes by \mathbf{C} , and the rationals by \mathbf{Q} . The symmetric group is denoted S_n and the cyclic group of order n is denoted C_n .

(20)1. Find an example of each of the following (no proof necessary):

- (a) A unique factorization domain that is not a principal ideal domain.
- (b) A field with exactly 4 subfields, including the field itself.
- (c) A subgroup of S_4 that is isomorphic to D_4 , the dihedral group of order 8.
- (d) A noncommutative ring R in which the only (two-sided) ideals are $\{0\}$ and R .

(10)2. Let G be a group of order n . Prove that G is isomorphic to a subgroup of $GL_n(\mathbf{C})$.

(10)3. Let G be a finite group and let p be the smallest prime dividing $|G|$. Prove that if H is a normal subgroup of G of order p , then H is contained in the center of G .

(10)4. Find, up to similarity, all the complex 4×4 matrices that satisfy the polynomial $(x - 2)^2(x - 3)$ but do not satisfy $(x - 2)^3$.

(10)5. Find two elements A, B in $GL_2(\mathbf{C})$ each of which has finite order but such that AB does not have finite order.

(10)6. Prove that every $n \times n$ complex matrix is similar to an upper triangular matrix. (You may not use Jordan canonical form.)

(10)7. (a) Find the number of subgroups of $C_5 \times C_5$, including $\{e\}$ and the whole group.

(b) Find the order of the centralizer of the element $(1, 2, 3)(4, 5, 6)$ in S_6 . (Recall that the centralizer of an element g in a group G is $\{x \in G | xg = gx\}$.)

(10)8. Let R be a commutative ring in which every nonzero ideal I , $I \neq R$, is maximal.

(a) Prove that R has at most 2 maximal ideals.

(b) Prove if R has exactly 2 maximal ideals, then there are fields F_1, F_2 , such that R is isomorphic to $F_1 \oplus F_2$.

(10)9. Let K be the finite field extension of \mathbf{Q} obtained by adjoining a root of the polynomial $x^6 + 3$.

(a) Prove that K contains a primitive 6-th root of unity.

(b) Prove that the polynomial $x^6 + 3$ factors into linear factors in $K[x]$.

Tier 1 Algebra Exam

August 2014

Be sure to justify all your answers. Remember to start each problem on a new sheet of paper. Each problem is worth 10 points.

- (1) Consider the matrix

$$M = \begin{bmatrix} -\frac{1}{2} + \mathbf{i} & 0 & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 2 & \frac{\sqrt{2}}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} + \mathbf{i} \end{bmatrix}$$

(where $\mathbf{i} = \sqrt{-1}$)

- (a) Calculate the eigenvalues of M .
- (b) Is M diagonalizable over \mathbb{C} ? Prove it is or explain why it is not.
- (c) Calculate the minimal polynomial of M .
- (2) (a) Let S_n denote the group of permutations of the set $\{1, 2, \dots, n\}$. How many different subgroups of order 4 does S_4 have? Justify your calculation. (Two subgroups are considered different if they are different as sets.)
- (b) There is a homomorphism of S_4 onto S_3 . (You do not need to prove that there exists such a homomorphism.) Show that there is no homomorphism of S_5 onto S_4 .
- (3) Let R be a commutative ring with unit and let $a, b \in R$ be two elements which together generate the unit ideal. Show that a^2 and b^2 also generate the unit ideal together.
- (4) Let \mathbb{F}_{p^n} denote the field with p^n elements and suppose that $p^n - 1 = q_1^{a_1} \cdots q_k^{a_k}$ for distinct primes q_i . Find the number of integers $r \in \{0, 1, \dots, p^n - 2\}$ for which the equation

$$x^r = a$$

has a solution for every $a \in \mathbb{F}_{p^n}$.

- (5) (a) Let G be a group and let H_1, H_2 be normal subgroups of G for which $H_1 \cap H_2 = \{e\}$. Assume that any $g \in G$ can be written $g = h_1 h_2$ with $h_i \in H_i$. Show that G is isomorphic to the direct product $H_1 \times H_2$.
- (b) Show by giving an example that the above conclusion can be false if you only assume that one of the H_i is normal.
- (6) (a) Find the degree of the splitting field of the polynomial $x^4 + 1$ over \mathbb{Q} .

- (b) Find the degree of the splitting field of the polynomial $x^3 - 7$ over \mathbb{Q} .
- (7) Show that a ring homomorphism $\phi : \mathbb{Q}[x] \rightarrow \mathbb{Q}[x]$ is an isomorphism if and only if $\phi(x) = ax + b$ for some $a, b \in \mathbb{Q}$, $a \neq 0$.
- (8) For V a vector space, $L : V \rightarrow V$ a linear map from V to itself, and a positive integer n , let $L^n = L \circ L \circ \cdots \circ L$ (n times).
- (a) Give an example of a pair $V, L : V \rightarrow V$ so that L is not the zero map, $L \neq Id$ and $L^2 = L$.
- (b) Give an example of a pair V, L so that $L \neq Id$ and $L^3 = Id$.
- (c) Prove that if V has finite dimension, then there exists an N so that $\ker L^n = \ker L^N$ for all $n \geq N$.
- (9) Consider the rings $\mathbb{F}_5[x]/(x^2)$, $\mathbb{F}_5[x]/(x^2 - 3)$, and $\mathbb{F}_5 \times \mathbb{F}_5$. Show that no two of them are isomorphic to each other.
- (10) (a) Show that any ring automorphism of \mathbb{R} sends every element of \mathbb{Q} to itself.
- (b) Show that any ring automorphism of \mathbb{R} sends positive numbers to positive numbers.
- (c) Deduce that \mathbb{R} has no nontrivial automorphisms.

ALGEBRA TIER 1

Each problem is worth 10 points.

- (1) Prove or give a counterexample: every $n \times n$ complex matrix A is similar to its transpose A^t .
- (2) Let M denote the 3×4 matrix

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 4 & 3 & 5 \\ 3 & 5 & 4 & 6 \end{pmatrix}.$$

Determine with proof the dimension of the space of 3×4 matrices N such that $N^t M = 0$.

- (3) Let V be a vector space and V_1, V_2, V_3 subspaces of V such that $\dim(V_i) = 2$ for all i and $\dim(V_i \cap V_j) = 1$ for all $i \neq j$. Prove that either $\dim(V_1 \cap V_2 \cap V_3) = 1$ or $\dim(V_1 + V_2 + V_3) = 3$.
- (4) Let $V = \mathbb{C}^2$. Let $T: V \rightarrow V$ denote a \mathbb{C} -linear transformation with determinant $a + bi$, $a, b \in \mathbb{R}$. Prove that if we regard V as a 4-dimensional real vector space, the determinant of T as an \mathbb{R} -linear transformation of this space is $a^2 + b^2$.
- (5) Let G be a finite group of order $n \geq 2$.
 - (a) Prove that G is always isomorphic to a subgroup of $\text{GL}_n(\mathbb{Z})$.
 - (b) Prove or disprove: G is always isomorphic to a subgroup of $\text{GL}_{n-1}(\mathbb{Z})$.
- (6) Prove that for any integer $n \geq 1$ and any prime $p \geq 2$, the symmetric group S_{np} contains an n -element subset P such that every non-trivial element of S_{np} of order p is conjugate to an element of P . Is there a set $P \subset S_{np}$ with the same property and less than n elements?
- (7) Let G be a group which is the union of subgroups G_1, G_2, \dots, G_n , $n \geq 2$. Show that there exists $k \in \{1, 2, \dots, n\}$ such that

$$\bigcap_{i \neq k} G_i \subseteq G_k.$$

- (8) Prove or give a counterexample:
 - (a) Let $f: R \rightarrow S$ be a ring homomorphism and let I be a maximal ideal of S . Then $f^{-1}(I)$ is maximal.

- (b) Let $f: R \rightarrow S$ be a ring homomorphism and let I be a maximal ideal of S . Then $f^{-1}(I)$ is prime.
- (9) Consider the ideal $I = (2, \sqrt{-10})$ of $\mathbb{Z}[\sqrt{-10}]$.
 - (a) Show I^2 is principal.
 - (b) Show I is not principal.
 - (c) Show $R/I \cong \mathbb{Z}/2\mathbb{Z}$ as abelian groups.
- (10) Are $\mathbb{F}_5[x]/(x^2 + 2)$ and $\mathbb{F}_5[y]/(y^2 + y + 1)$ isomorphic rings? If so, write down an explicit isomorphism. If not, prove they are not.

Tier I Algebra Exam

August, 2015

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- **Be sure to fully justify all answers.**
 - **Notation.** The sets of integers, rational numbers, real numbers, and complex numbers are denoted \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} , respectively. All rings are understood to have a unit and ring homomorphisms to be unit preserving.
 - **Scoring.** Each problem is worth 10 points. Partial credit is possible. Answers may be graded on clarity as well as correctness.
 - **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number and your test number on each sheet of paper.**
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1. Let V be the subspace of \mathbb{R}^4 spanned by $(1, 1, 1, 1)$, $(2, 1, 0, 1)$, and $(0, 1, 1, 2)$. Let W be the subspace of \mathbb{R}^4 spanned by $(0, -1, -1, -3)$ and $(1, 0, 0, 1)$. Find a basis for the subspace $V \cap W$ of \mathbb{R}^4 .
2. Find a matrix M such that
$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{pmatrix}.$$
3. Find up to similarity all matrices over a field \mathbb{F} having characteristic polynomial $(x - 3)^5(x - 4)^3$ and minimal polynomial $(x - 3)^2(x - 4)$.
4. Find (with justification) an element of largest order in the symmetric group S_{10} .
5. In the group $G := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, let H be the subgroup generated by $(0, 6, 0, 0)$, $(0, 0, 3, 1)$, and $(0, 1, 0, 1)$. By the Fundamental Theorem of Finitely Generated Abelian Groups, there exists an isomorphism between G/H and a product of cyclic groups. Find such an isomorphism explicitly.
6. Show that a finite group which is generated by two distinct elements each of order 2 must be isomorphic to a dihedral group.
7. Find the inverse of the unit 201 in the ring $\mathbb{Z}/2015\mathbb{Z}$.

(Continues on other side.)

8. Consider the ring $R = \mathbb{Q}[x, \sqrt{x}, \sqrt[4]{x}, \sqrt[8]{x}, \dots]$ consisting of finite sums of the form $\sum_{i=1}^m a_i x^{n_i}$ with $a_i \in \mathbb{Q}$ and n_i a positive rational number whose denominator is a power of two. Show that every finitely generated ideal in R is principal. Exhibit a nonprincipal ideal in R .
9. For a prime number p , let \mathbb{F}_p denote the field with p elements, and consider the ring $\mathbb{F}_p[x]/(x^3 - 1)$. Identify the group of units as a product of cyclic groups for:
- (a) $p = 29$
 - (b) $p = 31$
- (You do not need to be explicit about the isomorphism with the product of cyclic groups.)
10. Show that the ring $A := \mathbb{C}[x, y]/(x^2 - y^2 - 1)$ is an integral domain. Further show that every nonzero prime ideal in A is maximal.

Algebra Tier 1

January 2016

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

\mathbf{F}_2 , \mathbf{F}_7 , \mathbf{Q} , and \mathbf{R} denotes the field with two elements, the field with seven elements, the field of rational numbers, and the field of real numbers respectively.

Problem 1. (a) Find the eigenvalues of the complex matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Find the eigenvectors of A .

(c) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Problem 2. Find the Jordan canonical form of the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Problem 3. Find the Jordan canonical form of the matrix

$$C = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem 4. Let V be a vector space of dimension 5 over \mathbf{F}_7 . Find the number of 2-dimensional subspaces of V .

Problem 5. Let G be a nonabelian group of order 27. Find the class equation for G .

Problem 6. Prove that a group of order 51 is cyclic.

Problem 7. What is the order of the automorphism group of a cyclic group G of order 100?

Problem 8. Factor the polynomial $x^8 - x$ in the ring $\mathbf{F}_2[x]$.

Problem 9. Find the minimal polynomial of $\sqrt[5]{2}$ over the field $\mathbf{Q}(\sqrt[3]{3})$.

Problem 10. Describe all prime ideals in the ring $\mathbf{R}[x]$.

ALGEBRA TIER 1

Each problem is worth 10 points.

- (1) Classify, up to isomorphism, all groups of order 24 which are quotient groups of \mathbb{Z}^2 .
- (2) If x, y are elements of G such that $(xy)^{11} = (yx)^{19} = 1$, then x and y are inverses of one another.
- (3) Prove that for all $n \geq 3$, the symmetric group S_n contains elements x and y of order 2 such that xy is of order n .
- (4) Let G be a non-trivial subgroup of the additive group \mathbb{R} of real numbers such that $\{x \in G \mid -1 < x < 1\} = \{0\}$. Prove that there exists $r \geq 1$ such that $G = \{nr \mid n \in \mathbb{Z}\}$.
- (5) Let V be an n -dimensional complex vector space, $T: V \rightarrow V$ a linear transformation, and $v \in V$ a vector. Prove that $v, Tv, T^2v, \dots, T^{n-1}v$ spans V if and only if $v, Tv, T^2v, \dots, T^{n-1}v$ is a basis of V .
- (6) Let A and B be $m \times n$ and $n \times m$ complex matrices respectively. Show that every non-zero eigenvalue of AB is a non-zero eigenvalue of BA .
- (7) If $M = (a_{i,j})_{1 \leq i,j \leq 3}$ is a 3×3 complex matrix such that M and $\bar{M} = (\bar{a}_{i,j})$ have the same characteristic polynomial, prove that M has a real eigenvalue.
- (8) Let $R = \{\frac{m}{2^n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$, where \mathbb{N} denotes the set of nonnegative integers. Prove that R is a subring of \mathbb{Q} . For every ideal I of R , prove that there exists an ideal J of \mathbb{Z} such that $I = \{\frac{m}{2^n} \mid m \in J, n \in \mathbb{N}\}$.
- (9) Prove that if K is any finite extension of \mathbb{Q} , then there exists an integer n and a maximal ideal \mathfrak{m} of the n variable polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$ such that $K \cong \mathbb{Q}[x_1, \dots, x_n]/\mathfrak{m}$.
- (10) Prove that if F is a finite field whose order is a power of 3, then F contains a square root of -1 if and only if it contains a 4th root of -1 .

ALGEBRA TIER I JAN 2017

Instructions. Each problem is worth **10** points. You have **4** hours to complete this exam.

- (1) (a) Prove or disprove that, if $A, B \subset V$ are subspaces of a finite-dimensional vector space V , then

$$\dim(A + B) = \dim(A) + \dim(B) - \dim(A \cap B)$$

where $A + B$ is the subspace spanned by the union of A and B .

- (b) Prove or disprove that, if $A, B, C \subset V$ are subspaces of a finite-dimensional vector space V , then

$$\begin{aligned} \dim(A + B + C) &= \dim(A) + \dim(B) + \dim(C) \\ &\quad - \dim(A \cap B) - \dim(B \cap C) - \dim(A \cap C) \\ &\quad + \dim(A \cap B \cap C). \end{aligned}$$

- (2) Find the number of two dimensional subspaces of $(\mathbb{Z}/p)^3$, where p is a prime.
- (3) Show that an element of $\text{GL}_2(\mathbb{Z})$ has order 1, 2, 3, 4, 6, or ∞ . Find elements of each of these orders.
- (4) Show the groups $\langle a, b \mid ababa = babab \rangle$ and $\langle x, y \mid x^2 = y^5 \rangle$ are isomorphic. Here, $\langle x_i, i \in I \mid r_j = s_j, j \in J \rangle$ stands for the quotient of the free group generated by $\{x_i, i \in I\}$ by the normal subgroup generated by the elements $r_j s_j^{-1}, j \in J$.
- (5) Suppose G is a group and $a \in G$ is an element so that the subset $S = \{gag^{-1} \mid g \in G\}$ contains precisely two elements. Prove that G contains a normal subgroup N so that $N \neq \{1\}$ and $N \neq G$.
- (6) Let $M : \mathbb{Z}^3 \rightarrow \mathbb{Z}^3$ be the homomorphism

$$M(a, b, c) = (2a + 4b - 2c, 2a + 6b - 2c, 2a + 4b + c)$$

Does the quotient group $\mathbb{Z}^3/M(\mathbb{Z}^3)$ have any elements of order 4? does it have any elements of infinite order? Justify your answer.

- (7) (a) Show that any group of order p^2 is abelian for any prime p .
 (b) Let G be a group of order 2873. It can be shown that G contains one normal subgroup of order 17 and another normal subgroup of order 169. Use this assertion (which you need not prove) to show that G is abelian.
- (8) How many invertible elements are there in the ring $\mathbb{Z}/105$? Find the structure of the group of invertible elements as an abelian group.
- (9) Let $\mathbb{M}_n(\mathbb{C})$ denote the ring of $n \times n$ -matrices with complex entries (for a fixed $n \geq 2$).
 (a) Show that there is no pair $(X, Y) \in \mathbb{M}_n(\mathbb{C}) \times \mathbb{M}_n(\mathbb{C})$ such that $XY - YX = \text{Id}_n$, where Id_n is the $n \times n$ -identity matrix.
 (b) Exhibit a pair $(X, Y) \in \mathbb{M}_n(\mathbb{C}) \times \mathbb{M}_n(\mathbb{C})$ such that $\text{Rank}(XY - YX - \text{Id}_n) = 1$. If no such pair exists, prove that this is indeed the case.
- (10) Determine the degree of the field extension $\mathbb{Q}(\sqrt{2} + \sqrt[3]{5})$ over \mathbb{Q} .

Tier 1 Algebra Exam—August 2017

Be sure to justify all your answers. Remember to start each problem on a new sheet of paper. Each problem is worth 10 points.

- (1) (a) Prove that if V is a finite dimensional vector space and $L: V \rightarrow V$ is linear and satisfies $L \circ L = 0$ then $\dim(\ker(L)) \geq \frac{1}{2} \dim(V)$.
- (b) Give an example of $L: V \rightarrow V$ where $L \circ L = 0$ and $\dim(\ker(L)) = \frac{1}{2} \dim(V)$.
- (c) Give an example of a linear $L': V \rightarrow V$ where $\dim(\ker(L')) = \dim(\ker(L' \circ L')) = \frac{1}{2} \dim(V)$.
- (2) Let A and B be two 3×3 matrices with entries in \mathbb{R} so that A 's minimal polynomial is $x^2 - 4$ and B 's minimal polynomial is $x + 2$. Show that $A - B$ is a singular matrix. What are the possible values of $\dim(\ker(A - B))$?
- (3) *True or False.* If a claim is true, say it is true—no proof needed. If it is false, give a counterexample and explain why it is a counterexample.
- (a) A normal subgroup of a normal subgroup of a group G is normal in G .
- (b) If K and L are normal subgroups of a group G , then

$$KL = \{gh \mid g \in K \text{ and } h \in L\}$$

is a normal subgroup of G .

- (c) All the finitely generated subgroups of the additive group of real numbers are cyclic.
- (d) For any group G , the set of elements of order two in G $\{g \in G \mid g^2 = e\}$ forms a subgroup of G .
- (e) If G is finite and the center of G , $Z(G)$, satisfies $G/Z(G)$ is cyclic, then G is abelian.
- (4) Consider the group $GL_2(\mathbb{C})$ of 2×2 invertible matrices with complex entries. Let

$$\alpha = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \text{ and } \beta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and let $G = \langle \alpha, \beta \rangle$ be the subgroup of $GL_2(\mathbb{C})$ which is generated by α and β .

- (a) How many elements are there in G ?
- (b) Is G abelian?
- (c) Prove that every subgroup of G is normal in G .

- (5) A group G is called *Hopfian* if every onto homomorphism (epimorphism) from G to G has to be an isomorphism.
- Show that finite groups are Hopfian.
 - Show that the finitely generated free abelian groups \mathbb{Z}^n are Hopfian.
 - Deduce carefully that all finitely generated abelian groups are Hopfian.
 - Give an example of a non-Hopfian abelian group, A . Construct an epimorphism showing that A is not Hopfian.
- (6) Let A be a commutative ring. Given any ideal I in A , We can define an ideal in $A[x]$, which we denote $I[x]$, as follows: $I[x] = \{\sum_{j=0}^n r_j x^j \mid 0 \leq n \in \mathbb{Z}, r_j \in I\}$. Note that $I[x]$ is the set of all polynomials in $A[x]$ all of whose coefficients are in I . One can easily show (and you may assume it is true for this problem) that $I[x]$ is an ideal in $A[x]$.
- Assume that \mathcal{M} is a maximal ideal in A . Is the ideal $\mathcal{M}[x]$ a maximal ideal in $A[x]$? Prove your answer.
 - Assume that \mathcal{P} is a prime ideal in A . Is the ideal $\mathcal{P}[x]$ a prime ideal in $A[x]$? Prove your answer.
- (7) Let A, B, C , and D be integral domains. If $A \times B$ is isomorphic to $C \times D$, prove that A is isomorphic to C or D .
- (8) (a) Which finite fields \mathbb{F}_q contain a primitive third root of unity (that is, a third root of unity other than 1)? Give a clear condition on q .
- (b) Deduce for which q the polynomial $x^2 + x + 1$ splits into linear terms in $\mathbb{F}_q[x]$. Use the quadratic formula when appropriate to find these linear terms. Explain over which fields it is not appropriate to use the quadratic formula.
- (c) Which finite fields \mathbb{F}_q contain a square root of -3 , and which do not? Make sure your answer covers all finite fields.
- (9) Let \mathbb{F}_2 be the field with two elements. Consider the following rings: $\mathbb{F}_2[x]/(x^2)$, $\mathbb{F}_2[x]/(x^2 + 1)$, $\mathbb{F}_2[x]/(x^2 + x)$, $\mathbb{F}_2[x]/(x^2 + x + 1)$. Which two of these rings are isomorphic to each other? Which of these rings are fields? Prove your answers.
- (10) Let p and q be distinct primes.
- Prove that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$.
 - What is the degree of $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$ over \mathbb{Q} ? Prove your answer in detail.

Algebra Tier 1

January 2018

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

\mathbf{Z} , \mathbf{Q} , \mathbf{R} , and \mathbf{C} denotes the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Problem 1. Find the Jordan canonical form of the complex matrix

$$A = \begin{bmatrix} 1 & -2 & 1/2 \\ 2 & -4 & 1 \\ 3 & -6 & 3/2 \end{bmatrix}$$

Problem 2. Find a matrix $A \in M_{3 \times 3}(\mathbf{R})$ of rotation of \mathbf{R}^3 by 120 degrees about the vector $[1, 1, 1]^t$.

Problem 3. Suppose that V is a vector space of dimension n and $T : V \rightarrow V$ is a linear transformation having n distinct eigenvalues. If $H \subset V$ is an m dimensional subspace of V and $T(H) \subset H$, let T' denote the restriction of T to H . Prove that T' has m distinct eigenvalues as a linear transformation $T' : H \rightarrow H$.

Problem 4. Let T_1 and T_2 be linear operators on \mathbf{C}^n , such that $T_1 \circ T_2 = T_2 \circ T_1$. Prove that there exists a nonzero vector in \mathbf{C}^n which is an eigenvector for T_1 and for T_2 .

Problem 5. Prove that a group of order 77 is cyclic.

Problem 6. Let G be the quotient of the abelian group $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ by the subgroup generated by the elements $(2, 1, 5)$, $(1, 2, 10)$, $(2, 1, 7)$. Write G as a direct sum of cyclic groups.

Problem 7. Let C_n denote the cyclic group of order n . Find the order of the automorphism group of the abelian group $G = C_5 \oplus C_5$.

Problem 8. Give an example of a nontrivial group G , such that its automorphism group $\text{Aut}(G)$ contains a subgroup G' , which is isomorphic to G .

Problem 9. Let S be the set of polynomials $p(t)$ in the ring $\mathbf{Z}[t]$ for which $p(1)$ is even. Is S an ideal, and if so, is it principal?

Problem 10. The polynomial $p(x) = x^3 + 2x + 1$ is irreducible in $\mathbf{Q}[x]$, and thus $\mathbf{Q}[x]/\langle p(x) \rangle = F$ is a field. Any element $z \in F$ can be expressed as $z = \alpha_0 + \alpha_1x + \alpha_2x^2$ for some $\alpha_i \in \mathbf{Q}$. Find the values of the α_i in the case that $z = 1/(x - 1)$.

Tier 1 Algebra Exam
August 2018

Each problem is worth 10 points.

1. Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

2. Let S_{11} be the symmetric group in 11 letters. Find (with a proof) the smallest positive integer N such that all elements of S_{11} have order dividing N . You may leave your answer as a product of factors.
3. (a) (3 points) State the definition of a finitely generated ideal in a commutative ring R .
- (b) (7 points) Let R be the ring of continuous functions on the unit interval $[0, 1]$. Construct (with proof) an ideal in R which is not finitely generated.
4. Let $n \geq 1$ be an integer and let λ be a complex number. Determine the rank of the following $(n + 1) \times (n + 1)$ matrix. (Caution: Your answer may depend on the value of λ .)

$$\begin{pmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^n \\ 2 & 1 & \lambda & \dots & \lambda^{n-1} \\ 2 & 2 & 1 & \dots & \lambda^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & 1 \end{pmatrix}$$

5. Let A and B be normal subgroups of a group G such that $A \cap B = 1$. Prove that G has a subgroup isomorphic to $A \times B$.
6. Let $F \subset K$ be fields. Suppose $\alpha, \beta \in K$ are algebraic over F . Let $f(x) \in F[x]$ be the irreducible polynomial of α over F , and let $g(x) \in F[x]$ be the irreducible polynomial of β over F . Suppose that $\deg f(x)$ and $\deg g(x)$ are relatively prime. Prove that $g(x)$ is irreducible in $F(\alpha)[x]$. (We denote by $F(\alpha)$ the subfield of K generated by F and

- α . The irreducible polynomial of α over F is often called the minimal polynomial of α over F .)
7. (a) (7 points) Let V be a finite dimensional vector space over a field and let $T : V \rightarrow V$ be a linear transformation from V to itself such that $T(V) = T(T(V))$. Prove that $V = \text{Ker}T \oplus T(V)$.
 - (b) (3 points) Give an example showing the conclusion may not hold if $T(T(V)) \neq T(V)$.
 8. Let G be a finite group whose order is a power of a prime integer. Let $Z(G)$ denote the center of G . Show that $Z(G) \neq \{e\}$.
 9. Let $F \subset K$ be fields such that K is algebraic over F . Prove that if R is a *subring* of K containing F , then R is a subfield of K . (Caution: the degree $[K : F]$ may be infinite.)
 10. Let R be an integral domain, and let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x]$$

be a polynomial with coefficients in R of degree $n \geq 1$. Suppose $P \subset R$ is a prime ideal such that $a_i \in P$ for all $i = 0, 1, \dots, n-1$. Prove that if $f(x)$ is reducible, then $a_0 \in P^2$. (Note that $P^2 \subset R$ is the ideal generated by all products $\{ab : a, b \in P\}$ and that the leading coefficient of $f(x)$ is 1.)

Algebra Tier 1

January 2019

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

\mathbf{Q} , \mathbf{R} , and \mathbf{C} denotes the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Problem 1. Find the Jordan canonical form of the complex matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2. Let A be a complex square matrix such that $A^n = I$ for some $n \geq 1$. Prove that A is diagonalizable

Problem 3. A 5×5 complex matrix A has eigenvalues 1 and 0. If the rank $\text{rk}(A^2) = 1$ find all possible Jordan canonical forms of A .

Problem 4. Prove that a group of order 35 is cyclic.

Problem 5. Find the order of the automorphism group of the abelian group $G = C_3 \oplus C_3 \oplus C_3$, where C_3 is a cyclic group of order 3.

Problem 6. Describe the conjugacy classes in the dihedral group D_6 . (D_6 has 12 elements.)

Problem 7. Prove that the polynomial ring $\mathbf{Q}[x, y]$ contains an ideal I which can be generated by 3 elements, but not by 2 elements.

Problem 8. Give an example of a polynomial $p(x) \in \mathbf{R}[x]$ such that the quotient ring $\mathbf{R}[x]/(p(x))$ is not a product of fields.

Problem 9. Determine which of the following ideals are prime ideals or maximal (or neither) in the polynomial ring $\mathbf{C}[x, y]$:

$$I_1 = (x), I_2 = (x, y^2), I_3 = (x - y, x + y), I_4 = (x - y, x^2 - y^2)$$

Problem 10. Prove that the quotient ring $K = \mathbf{Q}[x]/(x^7 - 5)$ is a field. Then show that the polynomial $t^3 - 2$ is irreducible in the polynomial ring $K[t]$.

Algebra Tier 1

August 2019

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

1. Suppose H_1 and H_2 are subgroups of a finite group G . Prove that

$$[G : H_1 \cap H_2] \leq [G : H_1][G : H_2],$$

with equality if and only if every element of G can be written $h_1 h_2$ for some $h_1 \in H_1$ and $h_2 \in H_2$. Do **not** assume that G is abelian.

2. Let G be a group and $G^2 < G$ the subgroup generated by all elements in G of the form g^2 . Show that G^2 is normal in G and G/G^2 is an abelian group in which every element other than the identity has order 2.
3. Let p and q be two distinct prime numbers. Find the minimal polynomial of $\sqrt{p} + \sqrt{q}$ over \mathbb{Q} , and prove that it is indeed the minimal polynomial of $\sqrt{p} + \sqrt{q}$.
4. a. Show that $\mathbb{Z}[x]/(x^2 + x + 1, 5)$ is a field. How many elements does it have?
b. Show that $\mathbb{C}[x, y]/(xy - 1)$ and $\mathbb{C}[t]$ are not isomorphic rings.

5. Prove that there is an isomorphism of rings

$$\mathbb{C}[x, y]/(x - x^3 y) \rightarrow \mathbb{C}[y] \oplus \mathbb{C}[u, 1/u],$$

where $\mathbb{C}[u, 1/u]$ is the ring of Laurent polynomials $\sum_{i=-m}^n a_i u^i$ ($m, n \geq 0$) with complex coefficients.

6. Let F be a field of characteristic $p > 0$, n a positive integer, and N a nilpotent $n \times n$ matrix with entries in F (this means $N^k = 0$ for some positive integer k). Prove that $I + N$ is invertible, that it is of finite order in $\text{GL}_n(F)$, and that the order is a power of p .
7. Let V be a finite-dimensional vector space and $T: V \rightarrow V$ a linear transformation. Prove that

$$\dim \ker T^2 \geq \frac{\dim \ker T + \dim \ker T^3}{2}.$$

8. Let \mathbb{F}_q be a field with q elements. Let

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}_q^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}.$$

How many vector subspaces of dimension 2 of V contain the vector $(1, 1, -1, -1)$?

9. Give an explicit formula for

$$\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}^n$$

in terms of the positive integer n .

Tier 1 Algebra — January 2020

All problems carry equal weight. All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page.

1. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(i) Find the eigenvectors of A .

(ii) Find A^{100} .

Remark: When writing down the entries of A^{100} use exponential notation, e.g., $2^{100} + 3^{100}$, or similar.

2. Let A be an $n \times n$ -matrix. Show that

$$\det(A + I) = \sum_B \det B,$$

where the sum is over all the principal minors of A .

[The principal minors are the matrices B obtained from A by removing rows $1 \leq i_1 < \dots < i_r \leq n$ and columns $i_1 < \dots < i_r$ indexed by the same numbers, where r runs through $\{0, \dots, n\}$, and the determinant of the "empty matrix" is 1.]

3. Let $\phi : \mathbb{C}^7 \rightarrow \mathbb{C}^7$ be a \mathbb{C} -linear endomorphism with $\dim(\ker(\phi^3)) = 5$ and $\phi^7 = 0$. What are the possible Jordan canonical forms of ϕ ?

4. Let V be a finite-dimensional vector space over a field k , and let $U \subsetneq V$ be a proper subspace. Suppose $\phi : V \rightarrow V$ is an endomorphism of V which has the following properties:

- $\phi|_U = 0$, i.e., $U \subset \ker(\phi)$;
- the induced map $\bar{\phi} : V/U \rightarrow V/U$, defined by $\bar{\phi}(v + U) := \phi(v) + U$, is the zero map.

(i) Show that $\phi^2 = \phi \circ \phi$ is the zero map.

(ii) Is ϕ necessarily zero too? If so, give a proof, if not, give a counterexample.

5. Let G be a subgroup of S_6 , the symmetric group on 6 elements. Suppose G contains a 6-cycle. Prove that G has a normal subgroup H of index 2.

[Hint: You may use the sign homomorphism.]

6. (i) Let G be a finite group. Show that the size (i.e., cardinality) of every conjugacy class in G divides the order of G .

(ii) Use (i) to determine all finite groups G such that G has exactly two conjugacy classes.

7. Determine if the following assertions are true or false. Justify why or give a counterexample.

(i) Every subgroup of index 3 in a group is normal.

(ii) Every group is isomorphic to a subgroup of a symmetric group.

(iii) If a homomorphism $\phi : G \rightarrow H$ is onto then there is a subgroup K of G with $H \cong K$.

(iv) $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$.

8. Show that the group R^* of invertible elements in the ring $R = \mathbb{Z}/105\mathbb{Z}$, is isomorphic to $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

[Recall that an element x of a commutative ring R is called *invertible*, if there is an element y such that $xy = 1_R$.]

9. Let $\omega = e^{2\pi i/5}$ be a 5th root of unity. The field $L = \mathbb{Q}(\omega)$ contains $K = \mathbb{Q}(\sqrt{5})$ because $\omega + \omega^{-1} = \frac{-1+\sqrt{5}}{2}$.

(i) Find the minimal polynomial for ω over \mathbb{Q} . (Argue from first principles. Do not use "well-known" facts about cyclotomic polynomials.)

(ii) Find the minimal polynomial for ω over $\mathbb{Q}(\sqrt{5})$.

10. Let P be a prime ideal in a commutative ring R with 1. Let

$$f(x) = x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n \in R[x]$$

be a non-constant monic polynomial with coefficients in R . Suppose that all coefficients a_1, \dots, a_n are in P , and that $f(x) = g(x)h(x)$, for some non-constant monic polynomials $g(x), h(x) \in R[x]$. Then show that the constant term of $f(x)$ is in P^2 , the ideal generated by all elements of the form ab , where $a, b \in P$.

Solutions

1. (i) The eigenvalues are obviously 1,2,3, and corresponding eigenvectors are easily determined to be $(1, 0, 0)^T$, $(1, 1, 0)^T$, $(1, 1, 1)^T$.

(ii) Put

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so that $P^{-1}AP = D := \text{diag}(1, 2, 3)$, hence $A = PDP^{-1}$. The inverse matrix P^{-1} is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and we compute $A^{100} = PD^{100}P^{-1} =$

$$\begin{aligned} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2^{100} & 3^{100} \\ 0 & 2^{100} & 3^{100} \\ 0 & 0 & 3^{100} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\ & = \begin{bmatrix} 1 & 2^{100} - 1 & 3^{100} - 2^{100} \\ 0 & 2^{100} & 3^{100} - 2^{100} \\ 0 & 0 & 3^{100} \end{bmatrix} \end{aligned}$$

2. Think of the first column as $(1, 0, \dots, 0)^T + (a_{11}, \dots, a_{n1})^T$, and apply the linearity of the determinant in the columns. For each of the resulting two determinants, use in the same fashion linearity in their second columns. Continuing this way we end up with a sum of 2^n determinants, each with a subset of the columns replaced by columns consisting of a single 1 on the diagonal and zeros elsewhere. Expand along these columns to get principal minors of A .

3. As $\phi^7 = 0$, zero is the only eigenvalue of ϕ and the JCF J of ϕ is thus a block-diagonal matrix with blocks of type

$$J_m = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots & 0 \\ \vdots & 0 & 0 & \dots & \dots & 0 \\ \vdots & \dots & & & 0 & 1 \\ 0 & \dots & \dots & \dots & 0 & 0 \end{bmatrix}$$

which has size m . The sizes of those blocks must add up to 7. The rank of J_m^k is easily seen to be $\max\{m - k, 0\}$. Since $\dim(\ker(\phi^3)) = 5$, we see that ϕ^3 has rank 2, and there must thus be blocks of size ≥ 4 .

- A block of size 4 contributes a rank of 1 to ϕ^3 . Therefore, an occurrence of block size 4 implies the existence of another block of size ≥ 4 , which is impossible.

- A block of size 5 contributes a rank of 2 to ϕ^3 – that’s what we want – and we have thus two possibilities for J : $J = \text{diag}(J_5, J_2)$ or $J = \text{diag}(J_5, J_1, J_1)$.

- Blocks of size ≥ 6 cannot occur because they would contribute a rank of ≥ 3 .

4. (i) Because $\bar{\phi}$ is assumed to be the zero homomorphism, we see that $\phi(v) \in U$ for all $v \in V$. But since $U \subset \ker(\phi)$ we find $\phi(\phi(v)) = 0$.

(ii) No, ϕ does not need to be zero: take the endomorphism of k^2 associated to the Jordan block J_2 , and let $U = ke_1$.

5. Let $g \in G$ be an element of order 6, e.g. $g = (123456)$. Then $g^3 = (14)(25)(36)$, which is an odd permutation, so g is odd.

This means the homomorphism

$$\sigma: G \rightarrow \{1, -1\}$$

given by $\sigma(h) = \text{sign}(h)$ is onto. Let $K \triangleleft G$ be the kernel of σ . Then $[G : K] = |G/K| = |\{1, -1\}| = 2$. So N has index 2, and is thus normal.

6. (i) If C_g is the conjugacy class of g , then the map $G \rightarrow C_g, x \mapsto xgx^{-1}$, is surjective. We have $xgx^{-1} = ygy^{-1} \Leftrightarrow y^{-1}x \in C_G(g)$, where $C_G(g)$ is the centralizer of g . The fiber of this map over xgx^{-1} is thus $xC_G(g)$, and hence $|C_g| = [G : C_G(g)]$. In particular, the order of each conjugacy class divides the order of G .

(ii) Suppose $n = |G|$. One conjugacy class is the set $\{e\}$, the set containing only the identity element of G . Since there are only two conjugacy classes, the second conjugacy class has order $n - 1$, and thus $(n - 1) \mid n$. This implies $n = 2$.

7. (i). Every subgroup of index 3 in a group is normal.

False. The subgroup $N = \{(1, 2), e\}$ is an index 3 subgroup of S_3 , the symmetric group on 3 elements. But $(1, 2, 3)(1, 2)(1, 3, 2) = (2, 3)$ so N is not normal.

(ii). Every group is isomorphic to a subgroup of a symmetric group.

True.

(iii). If a homomorphism $\varphi: G \rightarrow H$ is onto then there is a subgroup K of G with $H \cong K$.

This is false. \mathbb{Z} is torsion free but has homomorphic image C_2 .

(iv). $C_6 \times C_6 \cong C_4 \times C_9$.

False. Every finitely generated abelian group can be uniquely written as a product of cyclic p -group factors. Since $C_6 \times C_6 \cong C_2 \times C_2 \times C_3 \times C_3$, it follows that $C_6 \times C_6 \not\cong C_4 \times C_9$.

8. As a ring, $\mathbb{Z}/105$ is $\mathbb{Z}/3 \times \mathbb{Z}/5 \times \mathbb{Z}/7$. Each of these three rings is a field, whose group of units is cyclic; thus

$$(\mathbb{Z}/105)^* = (\mathbb{Z}/3)^* \times (\mathbb{Z}/5)^* \times (\mathbb{Z}/7)^* = \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/6,$$

or, in canonical form, $\mathbb{Z}/12 \times \mathbb{Z}/2 \times \mathbb{Z}/2$.

9. (i) We have $1 + \omega + \omega^2 + \omega^3 + \omega^4 = \frac{\omega^5 - 1}{\omega - 1} = 0$, and $f(T) = 1 + T + T^2 + T^3 + T^4$ has ω as a root. This means that $[L : \mathbb{Q}] \leq 4$. Since $L \subset K$, and $[K : \mathbb{Q}] = 2$, we have $[L : \mathbb{Q}] \in \{2, 4\}$. Now, $[L : \mathbb{Q}] = 2$ would imply that $L = K$, but $K \subset \mathbb{R}$, and L is not contained in \mathbb{R} . Hence $[L : \mathbb{Q}] = 4$ and $f(T)$ must be the minimal polynomial.

Grading decision to be made: It's up to you to decide if you want to let the student use that $[K : \mathbb{Q}] = 2$ **without comment**, or if you want to see an argument like " $\sqrt{5}$ is not in \mathbb{Q} because $T^2 - 5$ is irreducible by Eisenstein". You may want to see such an argument and award 1 point for it.

(ii) This can be done in various ways. I would start by looking at the equation $\omega + \omega^{-1} = \frac{-1 + \sqrt{5}}{4} =: c \in K$ which we gave the student. Then multiply both sides with ω so as to get $\omega^2 + 1 - c\omega = 0$, then so $g(\omega) = 0$ with $g(T) = T^2 - cT + 1$. Because $[L : K] = 2$, the polynomial g must be the minimal polynomial of ω over K .

10. Let $S = R/P$ be the quotient ring. It is an integral domain. Denote the images in $S[x]$ of the polynomials $f, g, h \in R[x]$ by $\bar{f}, \bar{g}, \bar{h}$. Then $\bar{f} = x^n$, by our assumption on the coefficients of f . recall that g and h were assume to be monic: $g(x) = x^m + \text{lower order terms}$ and $h(x) = x^k + \text{lower order terms}$. Hence $m + k = n$. Moreover, $\bar{g}\bar{h} = x^n$.

It follows that \bar{g} and \bar{h} must be monomials $\bar{g} = x^m, \bar{h} = x^k$.

To see this, write $\bar{g}(x) = x^m + \dots + \bar{b}_i x^{m-i}$ and $\bar{h}(x) = x^k + \dots + \bar{c}_j x^{k-j}$ with non-zero $\bar{b}_i, \bar{c}_j \in S$. Suppose on the contrary that $i > 0$ or $j > 0$ (or both). Then

$$\bar{g}(x)\bar{h}(x) = x^{m+k} + \dots + \overline{b_i c_j} x^{m-i+k-j}$$

Since S is an integral domain, $\overline{b_i c_j} \neq 0$. But then $\bar{g}\bar{h}$ is not a monomial (since $m+k-i-j < m+k$). This shows that indeed \bar{g} and \bar{h} are both monomials.

Grading decision to be make: I leave it up to you if you want to see the argument explained in the previous paragraph with all the details given here.

Hence all but the leading coefficients of g and h are in P . In particular, their constant coefficients are in P . Thus we have $f(0) = g(0)h(0) \in P^2$.

Tier 1 Algebra Exam
August 2020

All your answers should be justified (except where specifically indicated otherwise): A correct answer without a correct proof earns little credit. Write a solution of each problem on a separate page. Each problem is worth 10 points.

1. Let $T : V \rightarrow W$ be a linear transformation between finite dimensional vector spaces. Show that there are bases \mathcal{B} of V and \mathcal{C} of W so that the matrix M of T relative to these bases has $M_{ii} = 1$ for $1 \leq i \leq \dim T(V)$ and all other entries zero.
2. Let H be the subgroup of \mathbb{Z}^2 generated by $\{(5, 15), (10, 5)\}$. Find an explicit isomorphism between \mathbb{Z}^2/H and the product of cyclic groups.
3. Let $\alpha, \beta \in \mathbb{C}$ be complex numbers with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3 = [\mathbb{Q}(\beta) : \mathbb{Q}]$. Determine the possibilities for $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$. Give an example, without proof, of each case.
4. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.
 - (a) Find a 2×2 matrix P such that PAP^{-1} is upper triangular.
 - (b) Find a formula for A^n .
5. Let H be a proper subgroup of a finite group G . Show that G cannot be the union of the conjugates of H , i.e., $G \neq \cup_{g \in G} gHg^{-1}$.
6. Let $R \subset S$ be commutative rings with 1. An element $u \in S$ is said to be integral over R if there is a monic polynomial $f(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_0 \in R[x]$ such that $f(u) = 0$. Show that if S is a field and every element of S is integral over R , then R is a field.
7.
 - (a) Find a 2×2 real matrix so that $A^5 = I$ but $A \neq I$.
 - (b) Show that there is no 2×2 integral matrix so that $A^5 = I$ but $A \neq I$.

- (c) Find a 4×4 integral matrix so that $A^5 = I$ but $A \neq I$.
8. Identify all isomorphism classes of groups of order 20 having a unique subgroup of order 5 and an element of order 4.
9. Let R be a commutative ring with 1. Let $U \subset R$ be a subset containing 1 such that $u \cdot v \in U$ for all $u, v \in U$. Let J be an ideal of R such that
- (a) $J \cap U = \emptyset$, and
 - (b) if I is an ideal of R strictly containing J , then $I \cap U \neq \emptyset$.

Show that J is a prime ideal.

10. Prove or give a counterexample.
- (a) Every element of a finite field is the sum of two squares.
 - (b) Every element of a finite field is the sum of two cubes.

Tier 1 Algebra exam – August 2021

All problems carry equal weight. All your answers should be justified. A correct answer without a correct proof earns little credit. Begin the solution of each problem on a new sheet of paper.

1. Let V be a finite vector space over the field \mathbb{R} of real numbers.

Suppose that $L : V \rightarrow V$ is an \mathbb{R} -linear map whose minimal polynomial (that is, the lowest degree monic polynomial which annihilates L) equals

$$t^3 - 2t^2 + t - 2.$$

Prove that there is a non-zero subspace $W \subset V$ such that $L^4|_W = \text{id}_W$, i.e., the restriction of $L^4 = L \circ L \circ L \circ L$ to W is the identity map.

2. (a) (5 pts) Let V be a finite-dimensional vector space over a field k which is not assumed to be algebraically closed. Let $L : V \rightarrow V$ be an endomorphism of V , and v_1, \dots, v_n eigenvectors of L with associated eigenvalues $\lambda_1, \dots, \lambda_n$ respectively. Assume that $\lambda_i \neq \lambda_j$ for all $i \neq j$. Prove that $\{v_1, \dots, v_n\}$ is a linearly independent set of vectors.

(b) (5 pts) Let k be a field and $a, b, c \in k$ fixed but arbitrary elements. Find the dimension of the kernel of the following map

$$k^3 \longrightarrow k^3, \quad (x_1, x_2, x_3) \mapsto (x_1 + ax_2, x_2 + bx_3, x_3 + cx_1).$$

Remark. Your answer will depend on a, b, c .

3. Let V be a vector space over a field k of dimension $n \geq 1$, let $\phi : V \rightarrow k$ be a *non-zero* linear function, and let $a \in V$ be a fixed *non-zero* vector. Consider the endomorphism $L_a : V \rightarrow V$ defined by $L_a(v) = v + \phi(v)a$.

- (i) Give the characteristic polynomial $\chi_{L_a}(t)$ of L_a .
- (ii) Determine the minimal polynomial of L_a (your answer may depend on $\phi(a)$).
- (iii) Identify the set of vectors $a \in V$ for which L_a is diagonalizable.

4. Let $L : \mathbb{C}^6 \rightarrow \mathbb{C}^6$ be such that $L^6 = 0$ and $\text{rank}(L^2) = 2$. Describe all possibilities for the Jordan canonical form of L .

5. Give explicit examples/descriptions of

- (a) two non-isomorphic *abelian* groups A, B of order 32, and
- (b) two non-isomorphic *non-abelian* groups G, H of order 32,

with complete arguments that A is not isomorphic to B and G is not isomorphic to H .

6. Given a *surjective* homomorphism of groups $\phi : G \rightarrow H$, define

$$\Gamma(\phi) = \{(g, \phi(g)) \mid g \in G\} \subset G \times H.$$

Prove that $\Gamma(\phi)$ is a subgroup of $G \times H$ and that $\Gamma(\phi)$ is a normal subgroup if and only if H is abelian.

7. Let G be a group of order p^4 for a prime number p with $|Z(G)| = p^2$. Calculate the number of conjugacy classes in G as a function of p .

8. Set $\omega = \sqrt{-6}$ and let $A = \mathbb{Z}[\omega] = \{a + b\omega \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$.

- (a) Define a surjective ring homomorphism $f : A \rightarrow \mathbb{Z}/5\mathbb{Z}$.
- (b) Let I be the kernel of f . Show that I is not a principal ideal, i.e., cannot be generated by one element.

Hint. For part (b) you may use the function $N(a + b\omega) = a^2 + 6b^2$, which is just the square of the complex absolute value.

9. (a) Let E be a finite field extension of F that is generated over F by a subset $S = \{\alpha_1, \dots, \alpha_n\}$ of E .

(a) Suppose $\alpha^2 \in F$ for all $\alpha \in S$. Show that the degree of the extension, $[E : F]$ (i.e. the dimension of E as an F vector space), is a power of 2.

(b) Suppose $\alpha^3 \in F$ for all $\alpha \in S$. Give an example for which $[E : F]$ is not a power of 3.

10. Let $\alpha = \sqrt[3]{2}$, $\beta = \sqrt[5]{2}$ be the positive third and fifth root of 2 in \mathbb{R} , respectively. Set $F = \mathbb{Q}(\alpha, \beta)$.

(a) Find $[F : \mathbb{Q}]$.

(b) Set $\gamma := \frac{\alpha^2}{\beta^3} \in F$ and show that $F = \mathbb{Q}(\gamma)$.

Algebra Tier 1

January 2022

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

\mathbf{Q} , \mathbf{R} , and \mathbf{C} denotes the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Problem 1. Find the Jordan canonical form of the complex matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2. Let A and B be $n \times n$ complex matrices such that $AB = BA$. Prove that A and B have a common eigenvector: there exists $0 \neq X \in \mathbf{C}^n$ such that

$$AX = \lambda X \quad \text{and} \quad BX = \mu X$$

for some $\lambda, \mu \in \mathbf{C}$.

Problem 3. Let \mathbf{F}_7 be the field with 7 elements, and consider the special linear group

$$SL_3(\mathbf{F}_7) = \{A \in M_{3 \times 3}(\mathbf{F}_7) \mid \det A = 1\}.$$

What is its order $|SL_3(\mathbf{F}_7)|$?

Problem 4. Give an example of 3 pairwise nonisomorphic groups of order 24.

Problem 5. Give an example of a group G that contains a conjugacy class with 5 elements.

Problem 6. Let G and H be finite groups of orders $|G| = 34$, $|H| = 100$. Prove that there exists a homomorphism

$$\phi : G \rightarrow H$$

such that $\ker(\phi) \neq G$.

Problem 7. Let \mathbf{F}_{11} be the field with 11 elements. Consider the quotient rings $A = \mathbf{F}_{11}[x]/(x^2 - 2)$ and $B = \mathbf{F}_{11}[x]/(x^2 - 5)$. Is there a ring homomorphism $\phi : A \rightarrow B$? Is there a ring homomorphism $\psi : B \rightarrow A$?

Problem 8. Give an example of two distinct polynomials $p(x) = x^2 + ax + b$ and $q(x) = x^2 + cx + d$ in $\mathbf{Q}[x]$ such that $p(x)$ and $q(x)$ are irreducible, and the quotient fields $\mathbf{Q}[x]/(p(x))$ and $\mathbf{Q}[x]/(q(x))$ are isomorphic.

Problem 9. Consider the polynomial ring $\mathbf{C}[x, y]$ with the ideal $I = (x^2 + y^2, x^2 - y^2 - 1)$. Prove that the quotient ring $\mathbf{C}[x, y]/I$ is not a field.

Problem 10. Prove that the quotient ring $K = \mathbf{Q}[x]/(x^8 - 5)$ is a field. Then show that the polynomial $t^3 - 7$ is irreducible in the polynomial ring $K[t]$.