Tier 1 Algebra – January 2020

All problems carry equal weight. All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page.

1. Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

   (i) Find the eigenvectors of A.

   (ii) Find A^{100}.

   Remark: When writing down the entries of A^{100} use exponential notation, e.g., $2^{100} + 3^{100}$, or similar.

2. Let A be an $n \times n$-matrix. Show that
$$\det(A + I) = \sum_B \det B,$$
   where the sum is over all the principal minors of A.

   [The principal minors are the matrices B obtained from A by removing rows $1 \leq i_1 < \ldots < i_r \leq n$ and columns $i_1 < \ldots < i_r$ indexed by the same numbers, where r runs through $\{0, \ldots, n\}$, and the determinant of the "empty matrix" is 1.]

3. Let $\phi : \mathbb{C}^7 \to \mathbb{C}^7$ be a be a $\mathbb{C}$-linear endomorphism with $\dim\Big(\ker(\phi^3)\Big) = 5$ and $\phi^7 = 0$. What are the possible Jordan canonical forms of ϕ?

4. Let V be a finite-dimensional vector space over a field k, and let $U \subsetneq V$ be a proper subspace. Suppose $\phi : V \to V$ is an endomorphism of V which has the following properties:

   - $\phi|_U = 0$, i.e., $U \subset \ker(\phi)$;
   - the induced map $\overline{\phi} : V/U \to V/U$, defined by $\overline{\phi}(v + U) := \phi(v) + U$, is the zero map.

   (i) Show that $\phi^2 = \phi \circ \phi$ is the zero map.

   (ii) Is ϕ necessarily zero too? If so, give a proof, if not, give a counterexample.

5. Let G be a subgroup of S_6, the symmetric group on 6 elements. Suppose G contains a 6-cycle. Prove that G has a normal subgroup H of index 2.

   [Hint: You may use the sign homomorphism.]

6. (i) Let G be a finite group. Show that the size (i.e., cardinality) of every conjugacy class in G divides the order of G.

   (ii) Use (i) to determine all finite groups G such that G has exactly two conjugacy classes.

7. Determine if the following assertions are true or false. Justify why or give a counterexample.

   (i) Every subgroup of index 3 in a group is normal.

   (ii) Every group is isomorphic to a subgroup of a symmetric group.

   (iii) If a homomorphism $\phi : G \to H$ is onto then there is a subgroup K of G with $H \cong K$.

   (iv) $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$.

8. Show that the group R^* of invertible elements in the ring $R = \mathbb{Z}/105\mathbb{Z}$, is isomorphic to $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   [Recall that an element x of a commutative ring R is called *invertible*, if there is an element y such that $xy = 1_R$.]

9. Let $\omega = e^{2\pi i/5}$ be a 5^{th} root of unity. The field $L = \mathbb{Q}(\omega)$ contains $K = \mathbb{Q}(\sqrt{5})$ because $\omega + \omega^{-1} = \frac{-1+\sqrt{5}}{2}$.

   (i) Find the minimal polynomial for ω over $\mathbb{Q}$. (Argue from first principles. Do not use "well-known" facts about cyclotomic polynomials.)

   (ii) Find the minimal polynomial for ω over $\mathbb{Q}(\sqrt{5})$.

10. Let P be a prime ideal in a commutative ring R with 1. Let

$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n \in R[x]$$

   be a non-constant monic polynomial with coefficients in R. Suppose that all coefficients $a_1, \ldots, a_n$ are in P, and that $f(x) = g(x)h(x)$, for some non-constant monic polynomials $g(x), h(x) \in R[x]$. Then show that the constant term of $f(x)$ is in P^2, the ideal generated by all elements of the form ab, where $a, b \in P$.