# Algebra Tier 1

August 2019

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

1. Suppose $H_1$ and $H_2$ are subgroups of a finite group $G$. Prove that
$$[G : H_1 \cap H_2] \leq [G : H_1][G : H_2],$$
with equality if and only if every element of $G$ can be written $h_1 h_2$ for some $h_1 \in H_1$ and $h_2 \in H_2$. Do **not** assume that $G$ is abelian.

2. Let $G$ be a group and $G^2 < G$ the subgroup generated by all elements in $G$ of the form $g^2$. Show that $G^2$ is normal in $G$ and $G/G^2$ is an abelian group in which every element other than the identity has order 2.

3. Let $p$ and $q$ be two distinct prime numbers. Find the minimal polynomial of $\sqrt{p} + \sqrt{q}$ over $\mathbb{Q}$, and prove that it is indeed the minimal polynomial of $\sqrt{p} + \sqrt{q}$.

4. a. Show that $\mathbb{Z}[x]/(x^2 + x + 1, 5)$ is a field. How many elements does it have?

   b. Show that $\mathbb{Z}[x, y]/(xy - 1)$ and $\mathbb{Z}[t]$ are not isomorphic rings.

5. Prove that there is an isomorphism of rings
$$\mathbb{C}[x, y]/(x - x^3 y) \to \mathbb{C}[y] \oplus \mathbb{C}[u, 1/u] \ ,$$
where $\mathbb{C}[u, 1/u]$ is the ring of Laurent polynomials $\sum_{i=-m}^{n} a_i u^i$ $(m, n \geq 0)$ with complex coefficients.

6. Let $F$ be a field of characteristic $p > 0$, $n$ a positive integer, and $N$ a nilpotent $n \times n$ matrix with entries in $F$ (this means $N^k = 0$ for some positive integer $k$). Prove that $I + N$ is invertible, that it is of finite order in $\mathrm{GL}_n(F)$, and that the order is a power of $p$.

7. Let $V$ be a finite-dimensional vector space and $T \colon V \to V$ a linear transformation. Prove that
$$\dim \ker T^2 \geq \frac{\dim \ker T + \dim \ker T^3}{2}.$$

8. Let $\mathbb{F}_q$ be a field with $q$ elements. Let
$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}_q^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}.$$
How many vector subspaces of dimension 2 of $V$ contain the vector $(1, 1, -1, -1)$?

9. Give an explicit formula for
$$\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}^n$$
in terms of the positive integer $n$.