Algebra Tier 1

January 2022

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

Q, **R**, and **C** denotes the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Problem 1. *Find the Jordan canonical form of the complex matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2. *Let A and B be $n \times n$ complex matrices such that $AB = BA$. Prove that A and B have a common eigenvector: there exists $0 \neq X \in \mathbf{C}^n$ such that*

$$AX = \lambda X \quad \text{and} \quad BX = \mu X$$

for some $\lambda, \mu \in \mathbf{C}$.

Problem 3. *Let $\mathbf{F}_7$ be the field with 7 elements, and consider the special linear group*

$$SL_3(\mathbf{F}_7) = \{A \in M_{3\times 3}(\mathbf{F}_7) \mid \det A = 1\}.$$

What is its order $|SL_3(\mathbf{F}_7)|$?

Problem 4. *Give an example of 3 pairwise nonisomorphic groups of order 24.*

Problem 5. *Give an example of a group G that contains a conjugacy class with 5 elements.*

Problem 6. *Let G and H be finite groups of orders $|G| = 34$, $|H| = 100$. Prove that there exists a homomorphism*

$$\phi : G \to H$$

such that $\ker(\phi) \neq G$.

Problem 7. *Let $\mathbf{F}_{11}$ be the field with 11 elements. Consider the quotient rings $A = \mathbf{F}_{11}[x]/(x^2 - 2)$ and $B = \mathbf{F}_{11}[x]/(x^2 - 5)$. Is there a ring homomorphism $\phi : A \to B$? Is there a ring homomorphism $: B \to A$?*

Problem 8. *Give an example of two* distinct *polynomials $p(x) = x^2 + ax + b$ and $q(x) = x^2 + cx + d$ in $\mathbf{Q}[x]$ such that $p(x)$ and $q(x)$ are irreducible, and the quotient fields $\mathbf{Q}[x]/(p(x))$ and $\mathbf{Q}[x]/(q(x))$ are isomorphic.*

Problem 9. *Consider the polynomial ring $\mathbf{C}[x, y]$ with the ideal $I = (x^2 + y^2, x^2 - y^2 - 1)$. Prove that the quotient ring $\mathbf{C}[x, y]/I$ is not a field.*

Problem 10. *Prove that the quotient ring $K = \mathbf{Q}[x]/(x^8 - 5)$ is a field. Then show that the polynomial $t^3 - 7$ is irreducible in the polymonial ring $K[t]$.*