Algebra Tier 1

January 2018

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

Z, **Q**, **R**, and **C** denotes the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Problem 1. *Find the Jordan canonical form of the complex matrix*

$$A = \begin{bmatrix} 1 & -2 & 1/2 \\ 2 & -4 & 1 \\ 3 & -6 & 3/2 \end{bmatrix}$$

Problem 2. *Find a matrix $A \in M_{3\times 3}(\mathbf{R})$ of rotation of $\mathbf{R}^3$ by 120 degrees about the vector $[1,1,1]^t$.*

Problem 3. *Suppose that V is a vector space of dimension n and $T : V \to V$ is a linear transformation having n distinct eigenvalues. If $H \subset V$ is an m dimensional subspace of V and $T(H) \subset H$, let T' denote the restriction of T to H. Prove that T' has m distinct eigenvalues as a linear transformation $T' : H \to H$.*

Problem 4. *Let T_1 and T_2 be linear operators on $\mathbf{C}^n$, such that $T_1 \circ T_2 = T_2 \circ T_1$. Prove that there exists a nonzero vector in $\mathbf{C}^n$ which is an eigenvector for T_1 and for T_2.*

Problem 5. *Prove that a group of order 77 is cyclic.*

Problem 6. *Let G be the quotient of the abelian group $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ by the subgroup generated by the elements $(2,1,5)$, $(1,2,10)$, $(2,1,7)$. Write G as a direct sum of cyclic groups.*

Problem 7. *Let C_n denote the cyclic group of order n. Find the order of the automorphism group of the abelian group $G = C_5 \oplus C_5$.*

Problem 8. *Give an example of a nontrivial group G, such that its automorphism group $Aut(G)$ contains a subgroup G', which is isomorphic to G.*

Problem 9. *Let S be the set of polynomials $p(t)$ in the ring $\mathbf{Z}[t]$ for which $p(1)$ is even. Is S an ideal, and if so, is it principal?*

Problem 10. *The polynomial $p(x) = x^3 + 2x + 1$ is irreducible in $\mathbf{Q}[x]$, and thus $\mathbf{Q}[x]/\langle p(x)\rangle = F$ is a field. Any element $z \in F$ can be expressed as $z = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ for some $\alpha_i \in \mathbf{Q}$. Find the values of the α_i in the case that $z = 1/(x-1)$.*