

Tier 1 Algebra Exam—August 2017

Be sure to justify all your answers. Remember to start each problem on a new sheet of paper. Each problem is worth 10 points.

- (1) (a) Prove that if V is a finite dimensional vector space and $L: V \rightarrow V$ is linear and satisfies $L \circ L = 0$ then $\dim(\ker(L)) \geq \frac{1}{2} \dim(V)$.
- (b) Give an example of $L: V \rightarrow V$ where $L \circ L = 0$ and $\dim(\ker(L)) = \frac{1}{2} \dim(V)$.
- (c) Give an example of a linear $L': V \rightarrow V$ where $\dim(\ker(L')) = \dim(\ker(L' \circ L')) = \frac{1}{2} \dim(V)$.
- (2) Let A and B be two 3×3 matrices with entries in \mathbb{R} so that A 's minimal polynomial is $x^2 - 4$ and B 's minimal polynomial is $x + 2$. Show that $A - B$ is a singular matrix. What are the possible values of $\dim(\ker(A - B))$?
- (3) *True or False.* If a claim is true, say it is true—no proof needed. If it is false, give a counterexample and explain why it is a counterexample.
- (a) A normal subgroup of a normal subgroup of a group G is normal in G .
- (b) If K and L are normal subgroups of a group G , then

$$KL = \{gh \mid g \in K \text{ and } h \in L\}$$

is a normal subgroup of G .

- (c) All the finitely generated subgroups of the additive group of real numbers are cyclic.
- (d) For any group G , the set of elements of order two in G $\{g \in G \mid g^2 = e\}$ forms a subgroup of G .
- (e) If G is finite and the center of G , $Z(G)$, satisfies $G/Z(G)$ is cyclic, then G is abelian.
- (4) Consider the group $GL_2(\mathbb{C})$ of 2×2 invertible matrices with complex entries. Let

$$\alpha = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \text{ and } \beta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and let $G = \langle \alpha, \beta \rangle$ be the subgroup of $GL_2(\mathbb{C})$ which is generated by α and β .

- (a) How many elements are there in G ?
- (b) Is G abelian?
- (c) Prove that every subgroup of G is normal in G .

- (5) A group G is called *Hopfian* if every onto homomorphism (epimorphism) from G to G has to be an isomorphism.
- Show that finite groups are Hopfian.
 - Show that the finitely generated free abelian groups \mathbb{Z}^n are Hopfian.
 - Deduce carefully that all finitely generated abelian groups are Hopfian.
 - Give an example of a non-Hopfian abelian group, A . Construct an epimorphism showing that A is not Hopfian.
- (6) Let A be a commutative ring. Given any ideal I in A , We can define an ideal in $A[x]$, which we denote $I[x]$, as follows: $I[x] = \{\sum_{j=0}^n r_j x^j \mid 0 \leq n \in \mathbb{Z}, r_j \in I\}$. Note that $I[x]$ is the set of all polynomials in $A[x]$ all of whose coefficients are in I . One can easily show (and you may assume it is true for this problem) that $I[x]$ is an ideal in $A[x]$.
- Assume that \mathcal{M} is a maximal ideal in A . Is the ideal $\mathcal{M}[x]$ a maximal ideal in $A[x]$? Prove your answer.
 - Assume that \mathcal{P} is a prime ideal in A . Is the ideal $\mathcal{P}[x]$ a prime ideal in $A[x]$? Prove your answer.
- (7) Let A, B, C , and D be integral domains. If $A \times B$ is isomorphic to $C \times D$, prove that A is isomorphic to C or D .
- (8) (a) Which finite fields \mathbb{F}_q contain a primitive third root of unity (that is, a third root of unity other than 1)? Give a clear condition on q .
- (b) Deduce for which q the polynomial $x^2 + x + 1$ splits into linear terms in $\mathbb{F}_q[x]$. Use the quadratic formula when appropriate to find these linear terms. Explain over which fields it is not appropriate to use the quadratic formula.
- (c) Which finite fields \mathbb{F}_q contain a square root of -3 , and which do not? Make sure your answer covers all finite fields.
- (9) Let \mathbb{F}_2 be the field with two elements. Consider the following rings: $\mathbb{F}_2[x]/(x^2)$, $\mathbb{F}_2[x]/(x^2 + 1)$, $\mathbb{F}_2[x]/(x^2 + x)$, $\mathbb{F}_2[x]/(x^2 + x + 1)$. Which two of these rings are isomorphic to each other? Which of these rings are fields? Prove your answers.
- (10) Let p and q be distinct primes.
- Prove that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$.
 - What is the degree of $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$ over \mathbb{Q} ? Prove your answer in detail.