# ALGEBRA TIER I JAN 2017

**Instructions.** Each problem is worth **10** points. You have **4** hours to complete this exam.

(1) (a) Prove or disprove that, if $A, B \subset V$ are subspaces of a finite-dimensional vector space $V$, then
$$\dim(A+B) = \dim(A) + \dim(B) - \dim(A \cap B)$$
where $A + B$ is the subspace spanned by the union of $A$ and $B$.

(b) Prove or disprove that, if $A, B, C \subset V$ are subspaces of a finite-dimensional vector space $V$, then
$$\begin{aligned}\dim(A+B+C) = \dim(A) + \dim(B) + \dim(C) \\ - \dim(A \cap B) - \dim(B \cap C) - \dim(A \cap C) \\ + \dim(A \cap B \cap C).\end{aligned}$$

(2) Find the number of two dimensional subspaces of $(\mathbb{Z}/p)^3$, where $p$ is a prime.

(3) Show that an element of $\mathrm{GL}_2(\mathbb{Z})$ has order 1, 2, 3, 4, 6, or $\infty$. Find elements of each of these orders.

(4) Show the groups $\langle a, b \,|\, ababa = babab \rangle$ and $\langle x, y \,|\, x^2 = y^5 \rangle$ are isomorphic. Here, $\langle x_i, i \in I \,|\, r_j = s_j, j \in J \rangle$ stands for the quotient of the free group generated by $\{x_i, i \in I\}$ by the normal subgroup generated by the elements $r_j s_j^{-1}, j \in J$.

(5) Suppose $G$ is a group and $a \in G$ is an element so that the subset $S = \{gag^{-1} \mid g \in G\}$ contains precisely two elements. Prove that $G$ contains a normal subgroup $N$ so that $N \neq \{1\}$ and $N \neq G$.

(6) Let $M : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the homomorphism
$$M(a, b, c) = (2a + 4b - 2c, 2a + 6b - 2c, 2a + 4b + c)$$
Does the quotient group $\mathbb{Z}^3/M(\mathbb{Z}^3)$ have any elements of order 4? does it have any elements of infinite order? Justify your answer.

(7) (a) Show that any group of order $p^2$ is abelian for any prime $p$.

(b) Let $G$ be a group of order 2873. It can be shown that $G$ contains one normal subgroup of order 17 and another normal subgroup of order 169. Use this assertion (which you need not prove) to show that $G$ is abelian.

(8) How many invertible elements are there in the ring $\mathbb{Z}/105$? Find the structure of the group of invertible elements as an abelian group.

(9) Let $\mathbb{M}_n(\mathbb{C})$ denote the ring of $n \times n$-matrices with complex entries (for a fixed $n \geq 2$).

(a) Show that there is no pair $(X, Y) \in \mathbb{M}_n(\mathbb{C}) \times \mathbb{M}_n(\mathbb{C})$ such that $XY - YX = \mathrm{Id}_n$, where $\mathrm{Id}_n$ is the $n \times n$-identity matrix.

(b) Exhibit a pair $(X, Y) \in \mathbb{M}_n(\mathbb{C}) \times \mathbb{M}_n(\mathbb{C})$ such that $\mathrm{Rank}(XY - YX - \mathrm{Id}_n) = 1$. If no such pair exists, prove that this is indeed the case.

(10) Determine the degree of the field extension $\mathbb{Q}(\sqrt{2} + \sqrt[3]{5})$ over $\mathbb{Q}$.