## Algebra Tier 1

## January 2016

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

 $\mathbf{F}_2$ ,  $\mathbf{F}_7$ ,  $\mathbf{Q}$ , and  $\mathbf{R}$  denotes the field with two elements, the field with seven elements, the field of rational numbers, and the field of real numbers respectively.

**Problem 1.** (a) Find the eigenvalues of the complex matrix

$$A = \left[ \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

- (b) Find the eigenvectors of A.
- (c) Find an invertible matrix P such that  $P^{-1}AP$  is diagonal.

**Problem 2.** Find the Jordan canonical form of the matrix

$$B = \left[ \begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right]$$

**Problem 3.** Find the Jordan canonical form of the matrix

$$C = \left[ \begin{array}{ccccc} 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array} \right]$$

**Problem 4.** Let V be a vector space of dimension 5 over  $\mathbf{F}_7$ . Find the number of 2-dimensional subspaces of V.

**Problem 5.** Let G be a nonabelian group of order 27. Find the class equation for G.

**Problem 6.** Prove that a group of order 51 is cyclic.

**Problem 7.** What is the order of the automorphism group of a cyclic group G of order 100?

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**Problem 8.** Factor the polynomial  $x^8 - x$  in the ring  $\mathbf{F}_2[x]$ .

**Problem 9.** Find the minimal polynomial of  $\sqrt[5]{2}$  over the field  $\mathbf{Q}(\sqrt[2]{3})$ .

**Problem 10.** Describe all prime ideals in the ring  $\mathbf{R}[x]$ .