Tier I Algebra Exam

August, 2015

- Be sure to fully justify all answers.
- Notation. The sets of integers, rational numbers, real numbers, and complex numbers are denoted \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} , respectively. All rings are understood to have a unit and ring homomorphisms to be unit preserving.
- Scoring. Each problem is worth 10 points. Partial credit is possible. Answers may be graded on clarity as well as correctness.
- Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number and your test number on each sheet of paper.
- 1. Let V be the subspace of \mathbb{R}^4 spanned by (1,1,1,1), (2,1,0,1), and (0,1,1,2). Let W be the subspace of \mathbb{R}^4 spanned by (0,-1,-1,-3) and (1,0,0,1). Find a basis for the subspace $V \cap W$ of \mathbb{R}^4 .
- 2. Find a matrix M such that

$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{pmatrix}.$$

- 3. Find up to similarity all matrices over a field \mathbb{F} having characteristic polynomial $(x-3)^5(x-4)^3$ and minimal polynomial $(x-3)^2(x-4)$.
- 4. Find (with justification) an element of largest order in the symmetric group S_{10} .
- 5. In the group $G := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, let H be the subgroup generated by (0,6,0,0), (0,0,3,1), and (0,1,0,1). By the Fundamental Theorem of Finitely Generated Abelian Groups, there exists an isomorphism between G/H and a product of cyclic groups. Find such an isomorphism explicitly.
- 6. Show that a finite group which is generated by two distinct elements each of order 2 must be isomorphic to a dihedral group.
- 7. Find the inverse of the unit 201 in the ring $\mathbb{Z}/2015\mathbb{Z}$.

(Continues on other side.)

- 8. Consider the ring $R = \mathbb{Q}[x, \sqrt{x}, \sqrt[4]{x}, \sqrt[8]{x}, \ldots]$ consisting of finite sums of the form $\sum_{i=1}^{m} a_i x^{n_i}$ with $a_i \in \mathbb{Q}$ and n_i a positive rational number whose denominator is a power of two. Show that every finitely generated ideal in R is principal. Exhibit a nonprincipal ideal in R.
- 9. For a prime number p, let \mathbb{F}_p denote the field with p elements, and consider the ring $\mathbb{F}_p[x]/(x^3-1)$. Identify the group of units as a product of cyclic groups for:
 - (a) p = 29
 - (b) p = 31

(You do not need to be explicit about the isomorphism with the product of cyclic groups.)

10. Show that the ring $A := \mathbb{C}[x,y]/(x^2-y^2-1)$ is an integral domain. Further show that every nonzero prime ideal in A is maximal.