ALGEBRA TIER 1

Each problem is worth 10 points.

(1) Prove or give a counterexample: every $n \times n$ complex matrix A is similar to its transpose A^t.

(2) Let M denote the 3×4 matrix

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 4 & 3 & 5 \\ 3 & 5 & 4 & 6 \end{pmatrix}.$$

Determine with proof the dimension of the space of 3×4 matrices N such that $N^t M = 0$.

(3) Let V be a vector space and V_1, V_2, V_3 subspaces of V such that $\dim(V_i) = 2$ for all i and $\dim(V_i \cap V_j) = 1$ for all $i \neq j$. Prove that either $\dim(V_1 \cap V_2 \cap V_3) = 1$ or $\dim(V_1 + V_2 + V_3) = 3$.

(4) Let $V = \mathbb{C}^2$. Let $T\colon V \to V$ denote a $\mathbb{C}$-linear transformation with determinant $a + bi$, $a, b \in \mathbb{R}$. Prove that if we regard V as a 4-dimensional real vector space, the determinant of T as an $\mathbb{R}$-linear transformation of this space is $a^2 + b^2$.

(5) Let G be a finite group of order $n \geq 2$.
   (a) Prove that G is always isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{Z})$.
   (b) Prove or disprove: G is always isomorphic to a subgroup of $\mathrm{GL}_{n-1}(\mathbb{Z})$.

(6) Prove that for any integer $n \geq 1$ and any prime $p \geq 2$, the symmetric group S_{np} contains an n-element subset P such that every non-trivial element of S_{np} of order p is conjugate to an element of P. Is there a set $P \subset S_{np}$ with the same property and less than n elements ?

(7) Let G be a group which is the union of subgroups $G_1, G_2, \ldots, G_n$, $n \geq 2$. Show that there exists $k \in \{1, 2, \ldots, n\}$ such that

$$\bigcap_{i \neq k} G_i \subseteq G_k.$$

(8) Prove or give a counterexample:
   (a) Let $f\colon R \to S$ be a ring homomorphism and let I be a maximal ideal of S. Then $f^{-1}(I)$ is maximal.

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   (b) Let $f\colon R \to S$ be a ring homomorphism and let I be a maximal ideal of S. Then $f^{-1}(I)$ is prime.

(9) Consider the ideal $I = (2, \sqrt{-10})$ of $\mathbb{Z}[\sqrt{-10}]$.
   (a) Show I^2 is principal.
   (b) Show I is not principal.
   (c) Show $R/I \cong \mathbb{Z}/2\mathbb{Z}$ as abelian groups.

(10) Are $\mathbb{F}_5[x]/(x^2+2)$ and $\mathbb{F}_5[y]/(y^2+y+1)$ isomorphic rings? If so, write down an explicit isomorphism. If not, prove they are not.