Tier 1 Algebra Exam

August, 2013

1. (8 points) Prove Fermat's Little Theorem: If p is a prime number and a is any integer, then $a^p - a$ is divisible by p.

2. (8 points) Compute A^{2013}, where $A = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$.

3. (8 points) Construct an explicit isomorphism between $\mathbb{Z}[\sqrt{-1}]/(7)$ and $\mathbb{Z}[\sqrt{-2}]/(7)$, where (7) denotes the ideal generated by 7.

4. (9 points) Let $\mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order $n > 1$.

   (a) Show that the automorphism group $A = Aut(\mathbb{Z}/n\mathbb{Z})$ is abelian.

   (b) What is the order of the automorphism group of the finite group $G = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/25\mathbb{Z}$?

   (c) Let G be as in part (b). Is the group $Aut(G)$ abelian?

5. (8 points) Show that $x^5 - (3+i)x + 2$ is irreducible in $(\mathbb{Z}[i])[x]$.

6. (8 points) For any pair of real numbers a and b, let $M_{a,b}$ be the $n \times n$ matrix

$$M_{a,b} = \begin{pmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a \end{pmatrix}$$

   with entries a on the diagonal and b off the diagonal. Find the eigenvalues of $M_{a,b}$ and their multiplicities.

7. (10 points) Classify (up to isomorphism) all groups of order 8. (You may use the following fact without proof: if $g^2 = 1$ for each element g in a group G, then G is abelian.)

8. (8 points) Let K be an algebraically closed field of characteristic $p > 0$, and let $q = p^n$. Show that the solutions of the equation $x^q = x$ form a subfield $F \subseteq K$.

9. (8 points) Let $M \in \mathcal{M}_n(\mathbb{C})$ be a diagonalizable complex $n \times n$ matrix such that M is similar to its complex conjugate $\overline{M}$; i.e., there exists $g \in GL_n(\mathbb{C})$ such that $\overline{M} = gMg^{-1}$. Prove that M is similar to a real matrix $M_0 \in \mathcal{M}_n(\mathbb{R})$.

10. (8 points) Let $p > 2$ be an odd prime. Let F be a field with $q = p^n$ elements. How many solutions of the equation

$$x^2 - y^2 = 1$$

are there with $x, y \in F$?

11. (8 points) Let G be a group. Let t be the number of subgroups of G that are not normal. Prove that $t \neq 1$.

12. (9 points) Let V be a vector space of dimension n over a finite field F with q elements.

    (a) Find the number of 1-dimensional subspaces of V.

    (b) Find the number of $n \times n$ invertible matrices with entries from F.

    (c) For each k, $1 \leq k \leq n$, find the number of k-dimensional subspaces of V.