ALGEBRA TIER I
August, 2012

All answers must be justified. A correct answer without justification will receive little credit. Each problem is worth 10 points.

(1) Let $f : \mathbb{Z}^3 \to \mathbb{Z}^4$ be the function

$$f(a,b,c) = (a+b+c, a+3b+c, a+b+5c, 4a+8b)$$

(a) Prove that $f$ is a group homomorphism.
(b) Let $H$ denote the image of $f$. Find an element of infinite order in $\mathbb{Z}^4/H$.
(c) Calculate the order of the torsion subgroup of $\mathbb{Z}^4/H$.

(2) Let $A$ be the matrix

$$A = \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

Calculate $[K : \mathbb{Q}]$ where $K$ is the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and the eigenvalues of $A$.

(3) Let $R = \mathbb{Z}[x]/I$, where $I$ is the ideal generated by $x^2 - 5x - 2$. Let $S$ denote the ring of $2 \times 2$ integer matrices: $S = M_2(\mathbb{Z})$. Let $B$ denote the matrix

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

(a) Show that there exists a unique ring homomorphism $f : R \to S$ satisfying $f(x + I) = B$.
(b) Let $J$ denote the ideal in $R$ generated by the element $(x-1) + I$. Is $f(J)$ an ideal in $S$? Why or why not?

(4) Does there exist a non-abelian group of order 2012?

(5) Let $n \in \{2, 3, 7\}$ and consider the ring

$$R_n = (\mathbb{Z}/n)[x]/(x^3 + x^2 + x + 2).$$

For which $n \in \{2, 3, 7\}$ (if any) is $R_n$ a field? For which $n$ (if any) is $R_n$ a integral domain but not a field? For which $n$ (if any) is $R_n$ not an integral domain? Justify all your conclusions.

(6) Let $R$ be the subring of $\mathbb{R}$ given by $R = \{n + m\sqrt{-10} \mid m, n \in \mathbb{Z}\}$. Show that the element $2 - \sqrt{-10}$ is irreducible in $R$ but not prime.

(7) Let $G$ be a finite abelian group and let $\phi : G \to G$ be a group homomorphism. Note that for all positive integers $k$ the function $\phi^k = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{k \text{ times}}$ is also a homomorphism from $G$ to $G$. Prove there is a positive integer $n$ such that $G \cong ker(\phi^n) \times \phi^n(G)$.

(8) Let $G$ be a group containing normal subgroups of order 3 and 5. Prove $G$ contains an element of order 15.

(9) Let $M$ be the following matrix:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ -4 & 1 & 0 \\ 3 & -2 & 3 \end{pmatrix}$$

Prove or disprove (by giving a counterexample) each of the following statements:

(a) For every $3 \times 4$ complex matrix $N$ there is a nonzero vector $v \in \mathbb{C}^4$ such that $MNv = 0$.

(b) For every $3 \times 4$ complex matrix $N$ there is a nonzero vector $v \in \mathbb{C}^3$ such that $NMv = 0$.

(10) Suppose that $K/F$ is a finite extension of fields and $p$ is the smallest prime dividing $[K : F]$. Prove that for all $\alpha \in K$, $F(\alpha) = F(\alpha^{p-1})$.