Algebra Tier 1

January 2012

All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page. The notation $\mathbf{F}_n, \mathbf{R}, \mathbf{Z}$ stands for the field with n elements, the field of real numbers, and the ring of integers respectively.

Problem 1. *Find the Jordan canonical form of the complex matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Problem 2. *Find the matrix A^{2011}, where A is the matrix from problem 1 above.*

Problem 3. *Find the eigenvalues and a basis for the eigenspaces of the matrix*

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 4. *Find the matrix $e^C := I + C + \frac{C^2}{2!} + \frac{C^3}{3!} + \ldots$, where*

$$C = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

Problem 5. *Let a and b be elements of a group G. Prove that ab and ba have the same order.*

Problem 6. *Prove that if G is a nonabelian group, then $G/Z(G)$ is not cyclic. Here $Z(G)$ denotes the center of G.*

Problem 7. *Prove that if G is a finite nonabelian group of order p^3, where p is a prime, then $Z(G) = [G, G]$, where $Z(G)$ denotes the center of G and $[G, G]$ denotes the commutator subgroup of G.*

Problem 8. *Let C_2 denote a cyclic group of order 2. Determine the group $Aut(C_2 \times C_2)$, calculating its order and identifying it with a familiar group.*

Problem 9. *Find all irreducible polynomials of degree ≤ 4 in $\mathbf{F}_2[x]$.*

Problem 10. *Find the set of polynomials in $\mathbf{F}_2[x]$ which are the minimal polynomials of elements in $\mathbf{F}_{16}$.*

Problem 11. *Prove that the rings $\mathbf{F}_{16}$, $\mathbf{F}_4 \times \mathbf{F}_4$, and $\mathbf{Z}/16\mathbf{Z}$ are pairwise non-isomorphic.*

Problem 12. *Find all the maximal ideals in the ring $\mathbf{R}[x]$.*

Problem 13. *Let R be the ring of Gaussian integers and $I \subset R$ be an ideal. If R/I has 4 elements what are the possibilities for I and R/I?*