Tier 1 Algebra Exam

August 2011

Do all 12 problems.

1. (8 points) Let A be a matrix in $GL_n(\mathbb{C})$. Show that if A has finite order (i.e., A^k is the identity matrix for some $k \geq 1$), then A is diagonalizable.

2. (8 points) Let V be a finite-dimensional real vector space of dimension n. Define an equivalence relation $\sim$ on the set $\mathrm{End}_{\mathbb{R}}(V)$ of $\mathbb{R}$-linear homomorphisms $V \to V$ as follows: if $S, T \in \mathrm{End}_{\mathbb{R}}(V)$ then $S \sim T$ if an only if there are invertible maps $A : V \to V$ and $B : V \to V$ such that $S = BTA$. (You may assume this is an equivalence relation.)

   Determine, as a function of n, the number of equivalence classes.

3. (8 points) Let $n \geq 2$. Let A be the n-by-n matrix with zeros on the diagonal and ones everywhere else. Find the characteristic polynomial of A.

4. (8 points) Find the Jordan canonical form of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}$.

   Justify your answer.

5. (8 points) Let $R = K[x, y, z]/(x^2 - yz)$, where K is a field. Show that R is an integral domain, but not a unique factorization domain.

6. (8 points) Let P be a prime ideal in a commutative ring R with 1, and let $f(x) \in R[x]$ be a polynomial of positive degree. Prove the following statement: if all but the leading coefficient of $f(x)$ are in P and $f(x) = g(x)h(x)$, for some non-constant polynomials $g(x), h(x) \in R[x]$, then the constant term of $f(x)$ is in P^2.

   [We recall that P^2 is the ideal generated by all elements of the form ab, where $a, b \in P$.h]

7. (10 points) Let p be a prime number and denote by $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field with p elements. For a positive integer n let $\mathbb{F}_{p^n}$ be the splitting field of $x^{p^n} - x \in \mathbb{F}_p[x]$. Prove that the following statements are equivalent:

   1) $k|n$.

   2) $(p^k - 1)|(p^n - 1)$.

   3) $\mathbb{F}_{p^k} \subset \mathbb{F}_{p^n}$.

8. (10 points) i) Show that $x^3 - 2$ and $x^5 - 2$ are irreducible over $\mathbb{Q}$.

   ii) How many field homomorphisms are there from $\mathbb{Q}[\sqrt[3]{2}, \sqrt[5]{2}]$ to $\mathbb{C}$?

   iii) Prove that the degree of $\sqrt[3]{2} + \sqrt[5]{2}$ over $\mathbb{Q}$ is 15.

9. (8 points) Let p be a prime number. Prove that any group of order p^2 is abelian.

10. (8 points) Let a be an element of a group G. Prove that a commutes with each of its conjugates in G if and only if a belongs to an abelian normal subgroup of G.

11. (8 points) Find the cardinality of $\mathrm{Hom}\,(\mathbb{Z}/20\mathbb{Z}, \mathbb{Z}/50\mathbb{Z})$, where $\mathrm{Hom}(\cdot, \cdot)$ denotes the set of group homomorphisms.

12. (8 points) Let G be a finite group, and let $M \subset G$ be a *maximal* subgroup, i.e., M is a proper subgroup of G and there is no subgroup M' such that $M \subsetneq M' \subsetneq G$. Show that if M is a normal subgroup of G then $|G : M|$ is prime.

    [Hint. Consider the homomorphism $G \to G/M$.]