Algebra Tier I Exam
August 2010

1. Find all irreducible monic quadratic polynomials in $\mathbb{Z}_3[x]$.
   (*Monic: coefficient of the highest power of x is one.*)

2. Let G be a finite group and $\Phi : G \to G$ an automorphism.

   (a) Show that Φ maps a conjugacy class of G into a conjugacy class of G.

   (b) Give a concrete example of non-trivial G and Φ such that $\{e\}$ is the only conjugacy class of G that Φ maps into itself. Explain.

   (c) Show that if $G = S_5$ (the symmetric group on five letters), then g and $\Phi(g)$ must be conjugate for any $g \in G$.

3. Let V and W be real vector spaces, and let $T : V \to W$ be a linear map. If the dimensions of V and W are 3 and 5, respectively, then for any bases B of V and B' of W, we can represent T by a 5×3 matrix $A_{T,B,B'}$. Find a set S of 5×3 matrices *as small as possible* such that for any $T : V \to W$ there are bases B of V and B' of W such that $A_{T,B,B'} \in S$.

4. Is it possible to find a field F with at most 100 elements so that F has exactly five different proper subfields? If so, find all such fields. If not, prove that no such field F exists.

5. Let G be the group of rigid motions (more specifically, rotations) in $\mathbb{R}^3$ generated by $x =$ a 90^o degree rotation about the x-axis, and $y =$ a 90^o degree rotation about the y-axis.

   (a) How many elements does G have?

   (b) Show that the subgroup generated by x^2 and y^2 is a normal subgroup of G.

6. In this problem, R is a finite commutative ring with identity. Define $a \in R$ to be *periodic of period k* if a, a^2, ..., a^k are all different, but $a^{k+1} = a$.

   (a) In $R = \mathbb{Z}_{76}$, find an element $a \neq 0, 1$ of period 1.

(b) In the same ring $R = \mathbb{Z}_{76}$ find an element that is not periodic.

(c) In $R = \mathbb{Z}_{76}$, list the possible periods and the number of elements of each period.

7. In this problem, R is a finite commutative ring with identity. Let $p(x) \in R[x]$, the ring of polynomials over R.

   (a) Show that $a \in R$ is a root of $p(x)$ if and only if $p(x)$ can be written as $p(x) = (x-a)g(x)$ with $g(x) \in R[x]$ of degree one less than the degree of $g(x)$.

   (b) Prove or give a counterexample: A polynomial of $p(x) \in R[x]$ of degree n can have at most n distinct roots in R.

8. Consider S_5, the symmetric group on 5 letters. If $\sigma \in S_5$ has order 6, how many elements of S_5 commute with σ?

9. Let A be a 5×5 *real* matrix of rank 2 having $\lambda = -i$ as one of its eigenvalues. Show that $A^3 = -A$ and that A is diagonalizable (as a complex matrix).

10. (a) Give an example of an irreducible monic polynomial of degree 4 in $\mathbb{Z}[x]$ that is reducible in the field $\mathbb{Q}[\sqrt{2}]$. Explain why your example has the stated properties.

    (b) Show that there is *no* irreducible monic polynomial of degree 5 in $\mathbb{Z}[x]$ that is reducible in the field $\mathbb{Q}[\sqrt{2}]$.

11. Let M be the ring of 3×3 matrices with integer entries. Find all maximal two-sided ideals of M.

12. For which values of n in $\mathbb{Z}$ does the ring $\mathbb{Z}[x]/(x^3 + nx + 3)$ have *no* zero divisors?