Tier I Algebra Exam
August, 2009

- **Be sure to fully justify all answers.**
- **Notation** The sets of integers, rational numbers, and real numbers are denoted $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$, respectively. For a prime integer p, $\mathbf{Z}/p$ denotes the quotient $\mathbf{Z}/p\mathbf{Z}$. All rings are understood to have a unit.
- **Scoring** Each problem is worth 10 points.
- **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**

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(1) Let A and B be finite subgroups of a group G of relatively prime orders; that is, $\gcd(|A|, |B|) = 1$. Prove that the function $\phi : A \times B \to G$ defined by $\phi(a, b) = ab$ is injective.

(2) Find an element of largest order in the symmetric group S_{12}. Justify your answer.

(3) (a) Let G and H be abelian groups and let $\mathrm{Hom}(G, H)$ denote the set of all group homomorphisms from G to H. If $\phi, \psi \in \mathrm{Hom}(G, H)$, define $\phi + \psi$ by $(\phi + \psi)(g) = \phi(g) + \psi(g)$ for all $g \in G$. Prove that $\mathrm{Hom}(G, H)$ is an abelian group.

(b) Let C be a cyclic group such that $\mathrm{Hom}(C, \mathbf{Z}/p) \cong \mathbf{Z}/p$. What can you say about the order of C?

(c) Let G be a finitely generated abelian group and $\mathrm{Hom}(G, \mathbf{Z}/p) \cong (\mathbf{Z}/p)^n$. What does this tell you about G?

(4) Let A, B be commuting 2×2 real matrices with characteristic polynomials $x^2 - 3x + 2$ and $x^2 - 1$, respectively. Show that either $A + B$ or $A - B$ has determinant 0.

(5) Suppose that T is a linear transformation of a finite dimensional real vector space V having characteristic polynomial $f(t)g(t)$ where f and g are relatively prime. Show that $V = \mathrm{Ker}\,(f(T)) \oplus \mathrm{Ker}\,(g(T))$.

(6) Let T be a linear transformation of a finite dimensional real vector space V and assume that V is spanned by eigenvectors of T. If $T(W) \subset W$ for some subspace $W \subset V$, show that W is spanned by eigenvectors. (Hint: consider the minimal polynomial of T.)

(7) Let F denote the field with two elements and let E be an extension field of F.

(a) Show that if $\alpha \in E$ satisfies $f(\alpha) = 0$ for some $f \in F[x]$, then $f(\alpha^2) = 0$.

(b) Suppose that $\alpha \in E$ is a root of the polynomial $f(x) = x^5 + x^2 + 1 \in F[x]$. List all roots of $f(x)$ in E in the form $a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4$ with each $a_i = 0$ or 1.

(8) Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be irreducible rational polynomials, having roots α and β, respectively. Find necessary and sufficient conditions on the coefficients (a, b, c, d) that imply that $\mathbf{Q}(\alpha)$ is isomorphic to $\mathbf{Q}(\beta)$. Prove that your conditions are both necessary and sufficient.

(9) (a) Is the ring $\mathbf{Z}[i]$ of Gaussian integers an integral domain? Justify your answer.

(b) Let $R = \mathbf{Z}[T]/\langle T^4 - 1\rangle$, where $\langle T^4 - 1\rangle$ is the ideal generated by $T^4 - 1$. Is R an integral domain? Justify your answer.

(c) Show that sending T to i determines a ring homomorphism $\psi : R \to \mathbf{Z}[i]$. Describe $\operatorname{Ker}\psi$ as all elements $a + bT + cT^2 + dT^3 \in R$ where a, b, c, d satisfy some conditions.

(10) Let F^* denote the multiplicative group of all nonzero elements of a finite field F. Show that F^* is cyclic.