TIER ONE EXAMINATION - ALGEBRA
AUGUST, 2008

Justify your answers. All rings are assumed to have an identity.
The numbers in parentheses are the points for that problem.

(9)1. Complete the following definitions:

(a) Let G be a group and let $g \in G$. The order of g is

(b) Let K/F be a field extension. An element $a \in K$ is called algebraic if

(c) An ideal I in a commutative ring R is called prime if

(12)2. Let G be a group and let $G_2 = \{g^2 \mid g \in G\}$. Let H denote the intersection of all subgroups of G containing G_2.

(a) Prove that H is a normal subgroup of G.

(b) Prove that G/H is abelian.

(c) Prove that if G/H is finite, its order is a power of 2.

(10)3. Let K be a field. Let $a, b \in K$ and let $R = K[x]/(x^2 + ax + b)$. Prove that exactly one of the following is true:

- R is a field.
- R is isomorphic to K^2, the direct sum of two copies of K.
- There is a nonzero element $r \in R$ such that $r^2 = 0$.

(8)4. A complex matrix A has characteristic polynomial $(x - 2)^4(x + 2)$ and minimal polynomial $(x - 2)(x + 2)$. Determine the possible Jordan canonical forms for A.

(12)5.Let V be an n–dimensional real vector space.

(a) Let a, b be nonnegative integers. Prove there are subspaces V_a and V_b of dimension a, b respectively with $V_a \cap V_b = 0$ if and only if $a + b \leq n$.

(b) Let a, b, c be nonnegative integers. Prove there are subspaces V_a, V_b and V_c of dimension a, b, c respectively with $V_a \cap V_b \cap V_c = 0$ if and only if $a \leq n$, $b \leq n$, $c \leq n$ and $a + b + c \leq 2n$.

(8)6. Let F be a field. Determine the possible finite groups G that are isomorphic to a subgroup of F^+, the additive group of F.

(10)7. A nonzero prime ideal P in a commutative ring R is called minimal if the only nonzero prime ideal Q contained in P is P itself. Now let F be a field and let $R = F[x, y]$, the polynomial ring in two variables over F. Prove that if P is a minimal prime ideal of R there is an irreducible element $f(x, y)$ in R such that $P = (f(x, y))$.

(10)8. Let D_n denote the dihedral group of order $2n$ (that is, D_n is the group of symmetries of the regular n–gon). Let G be a finite group. Prove that if there is a nontrivial homomorphism from D_n to G then the order of G is even.

(10)9. Let G be a group and let M, N be normal subgroups such that $MN = G$ and $M \cap N = \{e\}$. Prove that G is isomorphic to the direct product $G/M \times G/N$.

(10)10. Let $M_n(\mathbf{Q})$ denote the ring of $n \times n$ matrices over the rationals. Let K be a subring of $M_n(\mathbf{Q})$ such that K is a field and K contains $\mathbf{Q}$. Prove that the degree $[K : \mathbf{Q}]$ is finite and $[K : \mathbf{Q}]$ divides n.