Tier I Algebra Exam
August, 2007

- **Be sure to fully justify all answers.**
- **Notation** The sets of integers, rational numbers, real numbers, and complex numbers are denoted $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$, respectively. All rings are understood to have a unit.
- **Scoring** Each single part problem is worth 10 points. Each part of a multiple part problem is worth 5 points. (eg. Problem 1 is worth 10 points, Problem 2 is worth 25 points.)
- **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**

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(1) Prove that for a group G and positive integer k, if G contains an index k subgroup, then the intersection of all index k subgroups of G is a normal subgroup.

(2) Let $F : G \to G$ be an endomorphism, that is, a homomorphism from the group G to itself. Let F^n denote the n–fold composition of F with itself, and let $K_n = \text{Kernel}(F^n)$.

(a) Show that $K_n \subseteq K_{n+1}$ for all n.

(b) Let $F : (\mathbf{Z}/16\mathbf{Z})^3 \to (\mathbf{Z}/16\mathbf{Z})^3$ be the endomorphism defined by $F(x, y, z) = (2z, 2x, 8y)$. For all $n \geq 1$, describe K_n as a direct sum of cyclic groups.

(c) Show that if F is an endomorphism of the symmetric group S_5, $K_{n+1} = K_n$ for all $n \geq 2$.

(d) Give an example of an endomorphism F of the symmetric group S_5 for which $K_2 \neq K_1$.

(e) Prove that for general G and F, if $K_n = K_{n+1}$, then $K_n = K_{n+i}$ for all $i \geq 0$.

(3) Let $S = \{(x, y) \mid 23x + 31y = 1\ ,\ x + y < 100\} \in \mathbf{Z}^2$. Find the element of S for which $x + y$ is as large as possible.

(4) Let $T : \mathbf{R}^3 \to \mathbf{R}^4$ and $S : \mathbf{R}^4 \to \mathbf{R}^1$ be linear transformations given by:

$$T(x, y, z) = (x + 2y + z, x - y + 4z, x - y + 4z, 2x + y + 5z)$$

$$S(x, y, z, w) = (x - y + 2z - w).$$

Find two sets of vectors in $\mathbf{R}^4$, $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ such that $\{\alpha_1, \ldots, \alpha_m\}$ is a basis of $\text{Im}(T)$ and $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\}$ is a basis for $\text{Ker}(S)$. Justify your answer.

(5) Let $T_1, T_2 : \mathbf{C}^3 \to \mathbf{C}^3$ be linear transformations. Show that if both linear transformations have minimal polynomials of degrees at most 2, then there is a vector that is an eigenvector for both T_1 and T_2.

(6) Let R be a commutative ring with unity and suppose that for every $r \in R$ there is an $n \geq 2$ so that $r^n = r$. Show that every prime ideal in R is maximal.

(7) Suppose that R is an integral domain. Is it possible that R contains additive subgroups isomorphic to $\mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/q\mathbf{Z}$ for p and q distinct primes? Justify your answer.

(8) Prove that the polynomial $2x^4 + x + 1 \in \mathbf{Q}[x]$ is irreducible. Justify all your work.

(9) Let F_q denote the finite field with q elements. Show that for any $a \in F_q$ the equation $x^n = a$ has a solution in F_q if n is relatively prime to $q - 1$.

(10) Let $p(t) = t^3 - 2 \in \mathbf{Q}[t]$. Let $\alpha = \sqrt[3]{2}$ be the real root of p and let β be a complex root of p. Determine if $\alpha \in \mathbf{Q}[\beta]$ and explain your answer.

(11) Let $d > 1$ and let $p(x)$ and $q(x)$ be relatively prime irreducible polynomials in $\mathbf{Q}[x]$ of degree d. Suppose $p(\alpha) = 0 = q(\beta)$ for some $\alpha, \beta \in \mathbf{C}$. It follows that $1 \leq [\mathbf{Q}(\alpha, \beta) : \mathbf{Q}(\alpha)] \leq d$.

(a) Find an example of a d, p, q, α, and β, so that $[\mathbf{Q}(\alpha, \beta) : \mathbf{Q}(\alpha)] = 1$.

(b) Find an example of a d, p, q, α, and β, so that $[\mathbf{Q}(\alpha, \beta) : \mathbf{Q}(\alpha)] = d$.