Tier 1 Examination - Algebra

August 23, 2006

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by $\mathbf{R}$ and the set of rational numbers by $\mathbf{Q}$.

(20) 1. Find an example of each of the following (no proof necessary):

(a) An infinite integral domain in which there are exactly 4 units.

(b) Two nonisomorphic nonabelian groups of order 12.

(c) A unique factorization domain with exactly one irreducible element (up to multiplication by a unit).

(d) An element of order 3 in $GL_2(\mathbf{Q})$.

(10)2. Find the sum of the reciprocals of the eigenvalues of the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

(10)3. Let n be a positive integer. Let $V_0, V_1, \ldots, V_{2n-1}$ be a sequence of finite dimensional vector spaces. For $i = 0, 1, \ldots, 2n$, let $T_i : V_i \to V_{i+1}$ be linear transformations, where by convention, $V_{2n} = V_0$ and $T_{2n} = T_0$. Suppose that for $i = 0, 1, \ldots, 2n-1$, we have

$$ker(T_{i+1}) = im(T_i)$$

Prove that

$$dim(V_0) + dim(V_2) + dim(V_4) + \cdots + dim(V_{2n-2}) = dim(V_1) + dim(V_3) + \cdots dim(V_{2n-1}).$$

(10)4. Let R be a ring with unit (possibly non-commutative). An element α in R is called *left quasi-invertible* if $1 - \alpha$ is left invertible, that is, if there exists $b \in R$ such that $b(1 - \alpha) = 1$. A subset of R is called left quasi-invertible if all of its elements are left quasi-invertible.

(a) Show that if α is in every maximal left ideal, then α is left quasi-invertible.

(b) Show that if the left ideal generated by α is left quasi-invertible, then α is contained in every maximal left ideal.

(15)5. Let R be a commutative ring. If I and J are ideals in R we define the product ideal to be $IJ = \{\sum_{k=1}^{n} x_k y_k \mid n \geq 1 \text{ and } x_k \in I, y_k \in J\}$ and we define the sum ideal to be $I + J = \{x + y \mid x \in I, y \in J\}$.

(a) Prove that IJ is an ideal in R.

(b) Prove that $IJ \subset I \cap J$ and give an example to show that equality does not always hold.

(c) Prove that if $I + J = R$ then $IJ = I \cap J$.

(10)6. (a) Let R be an integral domain containing a subring F such that F is a field and such that R is finite dimensional as a vector space over F. Show that R is a field.

(b) Let T be a field extension of the field F and let K and L be intermediate fields such that K and L are both finite dimensional over F. Let $KL = \{\sum_{k=1}^{n} x_k y_k \mid n \geq 1 \text{ and } x_k \in K, y_k \in L\}$. Prove KL is a subfield of T.

(10)7. Let H and K be subgroups of the group G. Prove that $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

(10)8. Let G be a group and x, y elements of order 2. Let H be the subgroup generated by x and y. Prove that the subgroup generated by xy is normal in H and has index two in H.

(10)9. Let F be a field, let $f(X)$ be a polynomial with coefficients in F, and let $R = F[X]/(f(X))$.

(a) Suppose F is the rational numbers and $f(X) = X^2 - 1$. Let α be the image of $a_0 + a_1 X + \cdots + a_n X^n$ in R (for $a_0, \ldots, a_n \in F$). Find concise necessary and sufficient conditions on $a_0, \ldots, a_n$ for α to be a unit.

(b) Let $f(X) = X^3 - 3X^2 - 1$. Show that if F is the real numbers, then R has zero divisors, but if F is the rational numbers, then R does not.