## ALGEBRA TIER I

January 2006

Unless stated otherwise, all your answers require justification. A correct answer without a correct proof earns little credit. All questions are worth the same number of points.

1. Find the eigenvalues of the complex matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2. Consider the complex matrices

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find characteristic polynomials of $A$ and $B$.

b) Does there exist an invertible matrix $P$ such that $PAP^{-1} = B$?

3. Consider the complex matrices

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find the ranks of $C$ and $D$.

b) Does there exist an invertible matrix $P$ such that $PCP^{-1} = D$?

4. Let $A$ be a $n \times n$ complex matrix such that $A^2$ is the identity matrix. Prove that there exists an invertible $n \times n$ matrix $Q$ such that the matrix $QAQ^{-1}$ is diagonal.

5. Let $A$ be an invertible matrix, and let $E$ be an upper triangular matrix with zeroes on the diagonal, that is: the $(i, j)$'th entry of $E$ is $0$ for all $j \leq i$. Assume that $AE = EA$. Show that the matrix $A + E$ is invertible.

6. Give an example or compute, without proof, each of the following:

a) All the abelian groups, up to isomorphism, of order 144 (list them).

b) An infinite group all of whose elements have finite order.

c) Groups $H \lhd K \lhd G$ ( $H$ is normal in $K$ and $K$ is normal in $G$ ) such that $H$ is not normal in $G$.

7. Let $G$ be a group, and let $H$ and $K$ be normal subgroups with $H \cap K = \{e\}$. If $h \in H$ and $k \in K$ then show that $hk = kh$.

8. Prove that the center of the finite symmetric group $S_n$ is $\{e\}$ for $n \geq 3$. (Recall that the center is the set of elements which commute with every element of the group.)

9. How many conjugacy classes are there in the symmetric group $S_5$ ?

10. Show that if $G$ has trivial center, then $G$ is isomorphic to a subgroup of the group of automorphisms of $G$. (An *automorphism* of $G$ is an isomorphism $\phi : G \to G$. Note that the set of automorphisms forms a group under composition.)

11. What are the units in the ring of Gaussian integers $\mathbb{Z}[i]$ ?

12. a) Let $S$ be a subset of the complex numbers $\mathbb{C}$. Show that the intersection of all the fields $F \subseteq \mathbb{C}$ which contain $S$ is a field.

b) Let $F_1, F_2$ be subfields of $\mathbb{C}$, and assume that $[F_1 : \mathbb{Q}] = [F_2 : \mathbb{Q}] = 2$ and $F_1 \neq F_2$. Let $F_3$ be the minimal subfield of $\mathbb{C}$ which contains $F_1 \cup F_2$. Show that $[F_3 : \mathbb{Q}] = 4$.

13. a) Let $F$ be a finite field of order $p^a$. Show that every non-zero element $x \in F$ satisfies $x^{p^a - 1} = 1$.

b) Explain why the polynomial $x^n - 1$ can have at most $n$ roots in $F$.

c) Show that the non-zero elements of $F$, endowed with $F$ 's multiplication, form a cyclic group of order $p^a - 1$.