Algebra Tier 1 Exam August 2005

Time: 4 hours Points: 100

The point value for each problem is indicated. For problems with multiple parts, each part carries an equal point value unless indicated otherwise. Please provide an explanation for your answer to each problem, unless indicated otherwise.

1. (*8 points*) Given finite groups G_1, G_2 and H, along with surjective homomorphisms $\phi_i : G_i \to H$, define
$$P = \{(g_1, g_2) \in G_1 \times G_2 \mid \phi_1(g_1) = \phi_2(g_2)\}.$$
   a. Show that P is a subgroup and that the homomorphism $f\colon P \to G_1$ given by $f(g_1, g_2) = g_1$ is surjective.

   b. Find a relation between the orders of G_1, G_2, H, and P.

2. (*8 points*)

   a. What is the largest order of a cyclic subgroup of the symmetric group S_7?

   b. How many cyclic subgroups are there of the order found in part (a)?

3. (*10 points*) Let $\mathbb{Z}_n$ denote the additive cyclic group of order n. Let G be the group $\mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{11}$ and let H be the subgroup $\{6g \mid g \in G\}$.

   a. Express the groups H and G/H as direct sums of cyclic groups.

   b. Is G isomorphic to $H \oplus G/H$?

4. (*10 points*) Let G be a group and H a subgroup of G. Suppose further that $g^2 \in H$ for all $g \in G$.

   a. Show that H is a normal subgroup of G.

   b. Give an example of a group G with a non-normal subgroup K such that $g^4 \in K$ for all $g \in G$.

5. (*6 points*) Find polynomials $u(x)$ and $v(x)$ in $\mathbb{Q}[x]$ such that
$$u(x)(x^3 - 5) + v(x)(x^2 + 2x + 3) = 1.$$

6. (*6 points*) Let R be any ring with 1 such that $r^2 = r$ for all $r \in R$. Show that the characteristic of R is 2 and that R is commutative.

7. Let D be an integral domain that is not a field.

   a. *(2 points)* Show that D has a nonzero proper ideal.

   b. *(6 points)* Show further that D has infinitely many distinct ideals.

8. *(6 points)* Consider two subspaces W_1, W_2 of $\mathbb{R}^3$, each spanned by 2 vectors: W_1 by $\{(1,2,1),(-1,-1,0)\}$ and W_2 by $\{(2,3,3),(1,1,-2)\}$. Find a basis for $W_1 \cap W_2$.

9. *(6 points)* Show that there does not exist a 4×4 matrix A with real entries which satisfies all of the following properties: (i) the characteristic polynomial is $(x+2)^4$, (ii) $(A+2I)^2 \neq 0$, and (iii) rank $(A+2I)^2 =$ rank $(A+2I)^3$.

10. *(8 points)* Let V be a vector space of dimension greater than 1 over the real numbers. Let T be a linear transformation of V to itself of rank 1. Show that the minimal polynomial of T is $x(x-a)$ for some real number a.

11. *(6 points)* Let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^5 + 1$. If $\alpha \neq -1$, show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.

12. a. *(6 points)* Construct a field of 27 elements and find the structure of its additive group.

    b. *(4 points)* State without proof the structure of the multiplicative group of nonzero elements in that field.

13. *(8 points)* For each of the following, construct an example. No explanation is required.

    (a) An infinite field of characteristic $p \neq 0$.

    (b) A field K and a proper subfield E, both of which are algebraically closed.