Tier 1 Algebra Exam

January 2003

1. Give an example of each of the following. (No proofs required.)
   (a) A square matrix with real coefficients which is not diagonalizable.
   (b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose minimal polynomial is $x^2 - 2x + 1$.
   (c) An element of the alternating group A_{10} of order 12.
   (d) Four nonisomorphic groups of order 66.
   (e) A commutative ring with a prime ideal which is not maximal.

2. Suppose $p_1, p_2, p_3,$ and p_4 are real polynomials of degree 3 or less. Determine whether either of the following two conditions implies that the set $\{p_1, p_2, p_3, p_4\}$ is linearly dependent in the vector space of real polynomials. In each case, offer a proof or a counterexample.
   (a) $p_i(0) = 1$ for all i.
   (b) $p_i(1) = 0$ for all i.

3. Let $T : U \to V$ be a linear transformation of vector spaces and let $W \subset V$ be a subspace. Prove that $T^{-1}(W)$ is a subspace of U and that
$$\dim(T^{-1}(W)) \geq \dim(U) - \dim(V) + \dim(W).$$

4. Construct a field F with 16 elements as a quotient of $\mathbb{F}_2[x]$ and find a polynomial in $\mathbb{F}_2[x]$ whose image $\alpha \in F$ satisfies $\alpha^3 = 1$ and $\alpha \neq 1$. Justify your answer.

5. Let R be an infinite ring and let $\varphi : M_2(\mathbb{Z}) \to R$ be a a surjective ring homomorphism. Show φ is an isomorphism.

6. Let H and K be subgroups of a group G. Show that $HK = \{hk : h \in H, k \in K\}$ is a subgroup if and only if $HK = KH$.

7. For each $(x, y) \in \mathbb{Z}^2$, let $\langle (x, y) \rangle$ denote the subgroup of $\mathbb{Z}^2$ generated by (x, y). Express $\mathbb{Z}^2 / \langle (x, y) \rangle$ as a direct sum of cyclic groups.

8. Let $\alpha \in \mathbb{C}$ be an algebraic number, i.e. a root of a nonzero polynomial with rational coefficients.

(a) Show that α is a root of a unique monic irreducible polynomial $f(x)$ with rational coefficients.

(b) Show $\mathbb{Q}[\alpha]$ is a field.

(c) Show that the degree of $f(x)$ equals $[\mathbb{Q}[\alpha] : \mathbb{Q}]$.

(d) Write down $f(x)$ when $\alpha = \sqrt{2} + \sqrt{3}$. Justify your answer.

9. Let $F[x]$ denote the polynomial ring over a field F. Let R denote the subring $F[x^2, x^3]$.

   (a) Show $R \subsetneqq F[x]$.

   (b) Show that the quotient field of R is $F(x)$.

   (c) Show that R is not a unique factorization domain.

10. For a 2×2 matrix A with real coefficients, define e^A to be the matrix

$$\sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

Assuming that above series converges and that for all 2×2 matrices B and C

$$Be^A C = \sum_{n=0}^{\infty} \frac{1}{n!} BA^n C,$$

compute e^A where $A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$.