Tier 1 Examination - Algebra

January 3, 2002

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by $\mathbf{R}$ and the set of rational numbers by $\mathbf{Q}$. The set of integers modulo n is denoted $\mathbf{Z}_n$. The order of a set S is denoted $|S|$.

(10)1. What is the order of the group of invertible 2×2 matrices with entries in $\mathbf{Z}_5$?

(10) 2. Let G and H be finite abelian groups of the same order 2^n. Prove that if for each integer m,

$$|\{x \in G|\ x^{2^m} = 1\}| = |\{x \in H|\ x^{2^m} = 1\}|,$$

then G and H are isomorphic.

(10)3. Prove or give a counterexample for each of the following:

(a) For every integer n there is a finite group that cannot be generated by n elements.

(b) If G and H are finite groups such that $|G|$ and $|H|$ are relatively prime, then there exists a unique homomorphism from G to H.

(c) Every quotient group H of a group G is isomorphic to a subgroup of G.

(10)4. Prove that there exist algebraic numbers α and β, each of degree 3, such that $[\mathbf{Q}(\alpha, \beta) : \mathbf{Q}] = 6$.

(10)5. Show that for every prime p there are exactly $\frac{p^3-p}{3}$ irreducible cubic polynomials with leading coefficient 1 over the field $\mathbf{Z}_p$.

(10)6. Prove or disprove: Every maximal ideal of the real polynomial ring $\mathbf{R}[x, y]$ is of the form $(x - a, y - b)$ for some $a, b \in \mathbf{R}$.

(10)7. Prove that for every positive integer $n \geq 6$ which is not prime, there exist integers a and b such that the congruence equation $x^2 + ax + b \equiv 0 \pmod{n}$ has more than two solutions modulo n.

(10)8. Let V be a finite dimensional complex vector space and $T : V \to V$ a linear transformation. Suppose there exists $v \in V$ such that $\{v, T(v), T^2(v), \ldots, T^{n-1}(v)\}$ is a basis for V. Show that the eigenspaces of T are all 1–dimensional.

(10)9. Prove that a 5×5 skew-symmetric matrix A has determinant 0. (Recall that a matrix is called skew-symmetric if the transpose of A is the negative of A.)

(10)10. For each of the following conditions on a complex square matrix M, determine whether the condition implies that M is diagonalizable.

(a) $M^2 = M$

(b) $M^3 = I$

(c) $M^4 = 0$