Algebra Tier 1 Examination

August 2001

Time: 4 hours

Notation:

$\Re$: field of real numbers $\Re^n$: Euclidean n-space $\mathbb{Z}$: ring of integers
$\mathbb{Z}_n$: ring of integers modulo n $\mathbb{Q}$: field of rational numbers

1. Give examples (no need to prove anything) OR give mathematical reasons if you can't give examples of the following: (20 points)

   (a) An integral domain which is not a U.F.D. (unique factorization domain)

   (b) A field with 6 elements.

   (c) A 5×3 matrix P and a 3×5 matrix Q (with entries in $\Re$) such that $P \cdot Q = I_5$ (identity matrix of size 5).

   (d) A prime number p such that $x^2 - x + 2$ divides $3x^3 - 7x^2 + 40x + 27$ when considered as elements of $\mathbb{Z}_p[x]$.

   (e) An element of order 15 in S_9.

2a. Let G be a finite group of order n. Let H be a unique subgroup of G of index h. Show that H is a normal subgroup. Give example of a finite group L and at least two of its normal subgroups M and N of the same index. (5 points)

2b. Find all elements in $\mathbb{Z}_7$ which are squares of other elements. Use this and the natural homomorphism $f : \mathbb{Z} \longrightarrow \mathbb{Z}_7$ to show that the arithmetic sequence 10, 17, 24, 31 .... has no perfect squares. (5 points)

2c. Describe all the abelian groups G of order 500 such that $g^{50} = e$ (identity of G) for all $g \in G$. (Show all the work.) (5 points)

3a. Let H and K be cyclic groups of order 6 and 4 respectively with generators a and b respectively. Let $G = H \times K$. Let L be the subgroup of G generated by (a^5, b^2). Find the order of the element $\alpha = (a, b) \cdot L$ in the factor group $G/_L$. Hence or otherwise conclude that $G/_L$ is cyclic. (8 points)

3b. Let G be a group of order 21 which contains unique subgroups of order 3 and 7. Find the number of elements of order 21 in G. (7 points)

4a. Let R be a commutative ring with 1. For any $x \in R$, let $I_x = \{r \in R \quad rx = 0\}$. Show that I_x is an ideal in R. Find this ideal for $R = \mathbb{Z}_{24}$ and $x = 15$. (5 points)

4b. Let R be a commutative ring with 1 which has exactly 3 ideals $\{0\}, J$ and R. If $a \in R$ is not a zero divisor then show that a is a unit. (Hint: If a is not a unit then show first that $R \cdot a = R \cdot a^2 = J$ and then get a contradiction. (8 points)

5a Show that $x^2 - 4x - 4$ is irreducible in $\mathbb{Z}_5[x]$. (2 points)

5b. Show that $R = \mathbb{Z}_5[x]/_I$ (where I is the ideal generated by $x^2 - 4x - 4$) is a field. (You may quote and use any relevant results). Find also the number of elements in R. (5 points)

5c. Let I, R be as in $\{5b\}$. Let $\alpha = x + I \in R$. For this element, find the additive order (i.e. order as an element of the additive group $(R, +)$) and the multiplicative order (i.e order as an element of the multiplicative group $(R - \{0\}, \cdot)$. (5 points)

5d. Find elements $a, b \in \mathbb{Z}_5$ such that $(2\alpha + 1)(a\alpha + b) = 1$. (5 points)

6a Let V be a real vector space and let $S = \{v_1, v_2, \cdots v_m\}$ be a maximal linearly independent set (i.e S is not properly contained in another linearly independent set. Show that S is a basis of V. (4 points)

6b. Let $V = \Re^3$. Let $W = \{(a, b, c) \in \Re^3 \mid a = 2b + 3c \text{ and } b^2 = ac\}$. Determine with justification if W is a subspace of V. (4 points)

7. Let V denote the vector space of polynomials of degree at most 1 with real coefficients. Consider the linear transformation $T : V \longrightarrow V$ given by $T(f(x)) = -f(1) + f(-3)x$ (where f(a) is the value of the polynomial f(x) at a).

(a) Find the matrix of T with respect to the basis $\{1, x\}$ of V. (3 points)

(b) Find the eigenvalues of T and a basis for each eigenspace. (5 points)

(c) Is T diagonalizable ? Justify your answer. (4 points)