Tier 1 Examination - Algebra

January 4, 2001

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by $\mathbf{R}$. The set of integers modulo n is denoted $\mathbf{Z}_n$. The group of permutations on n letters is denoted S_n.

(20)1. Give an example of each of the following. No justification is required.

(a) A nonabelian group of order 18.

(b) An infinite commutative ring R such that for all $y \in R$, $y + y + y = 0$.

(c) A 3 by 3 real matrix that is diagonalizable over the complex numbers but not over the reals.

(d) A unique factorization domain that is not a principal ideal domain.

(e) An element of order 3 in $\mathbf{GL}_2(\mathbf{R})$.

(10) 2. Let G be a group with the property that $g^2 = e$ for all $g \in G$. Prove G is abelian.

(10)3. Determine the number of homomorphisms from S_3 to $\mathbf{Z}_2 \times \mathbf{Z}_4$.

(10)4. Find $lim_{n\to\infty} \begin{pmatrix} 2 & 3 \\ -1/2 & -1/2 \end{pmatrix}^n$.

(10)5. Let n be a positive odd integer and let $A, B \in M_n(\mathbf{R})$ such that $A^2 = B^2 = I$. Prove that A and B have a common eigenvector (not necessarily with the same eigenvalue).

(10)6. Let R and S be commutative rings and let $\phi : R \to S$ be a ring homomorphism. Suppose there is an ideal I of R such that $\ker(\phi) \subset I \subset R$ (proper containments). Prove that the image of ϕ is not a field.

(10)7. Let F be a subfield of $\mathbf{R}$ and suppose m and n are positive integers with $\sqrt{m}+\sqrt{n} \in F$. Prove that $\sqrt{m}$ and $\sqrt{n}$ are in F.

(10)8. Let K be a field and let $K^\times$ denote the group of nonzero elements of K. Prove that $K^\times$ contains at most two elements of order 6.

(10)9. Let $G = (\mathbf{Q}, +)/(\mathbf{Z}, +)$, where $(\mathbf{Q}, +)$ denotes the group of rational numbers under addition and $(\mathbf{Z}, +)$ denotes the subgroup of integers under addition. Prove that G is an infinite group in which every element has finite order.

(15)10. (a) Let R be a commutative ring and suppose I and J are ideals of R such that $I + J = R$, where $I + J = \{i + j | i \in I, j \in J\}$. vProve the map $\phi : R/I \cap J \to R/I \times R/J$ given by $\phi(r + I \cap J) = (r + I, r + J)$ is an isomorphism of rings.

(b) Let R be a commutative ring containing exactly 4 ideals (including $\{0\}$ and R). Let I and J denote the other two ideals and suppose they are incomparable, that is $I \not\subseteq J$ and $J \not\subseteq I$. Prove that R is isomorphic to the direct product of two fields.