Tier One Algebra Exam

August, 2000

1. (14 points) Let A be the 3×3 matrix all of whose entries are 1, i.e.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(i) Find the characteristic polynomial of A.

(ii) Find the minimal polynomial of A.

(iii) Find all eigenvalues of A.

(iv) Is A diagonalizable? If the answer is yes, find P such that PAP^{-1} is diagonal. If the answer if no, provide a reason.

2. (8 points) Let T and S be linear transformations from R^n to R^m. The coincidence set for T and S is defined to be the set $C(T, S) = \{w \in R^m \mid T(x) = w = S(y),$ for some $x, y \in R^n\}$. Let T and S be the linear transformations from R^4 to R^3 represented by the 3×4 matrices

$$T = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 5 & 4 & 1 & 0 \\ 3 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 2 \\ -1 & 4 & 11 & 6 \end{pmatrix}$$

Find a basis for $C(T, S)$.

3. (12 points) Let G be a group.

(i) Prove that the intersection of a family $F = \{H_t : t \in T\}$ of subgroups of G is a subgroup.

(ii) Let H be a subgroup of G. Prove that the intersection $K = \cap\{gHg^{-1} : g \in G\}$ of all conjugates of H is a normal subgroup.

(iii) Let H and K be as in (ii). If $[G : H] < \infty$, prove that $[G : K] < \infty$.

4. (6 points) Give an example of a nonabelian group with every proper subgroup abelian.

5. (6 points) Let S_{10} denote the symmetric group in 10 letters. Find the smallest n such that $a^n = 1$, for all $a \in S_{10}$. Justify your answer.

6. (10 points)

(i) Let G be an abelian group of order mn, where m and n are relatively prime. Let $H := \{g \in G : |g| \text{ divides } m\}$ and $K := \{g \in G : |g| \text{ divides } n\}$. Prove that the homomorphism $H \times K \to G$ induced by inclusion is an isomorphism.

(ii) Let G be an abelian group of order $p_1^{e_1} \cdots p_t^{e_t}$, where p_i's are distinct primes. For each prime p dividing $|G|$, let $G_p := \{g \in G : |g| = p^n \text{ for some } n\}$. Prove that $G \cong G_{p_1} \times \cdots \times G_{p_t}$.

7. (12 points)

(i) Let p and q be distinct primes. Let $n = pq$. Prove that $\mathbb{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}_q$ as rings.

(ii) Let n, p, q be as in (ii). Let ℓ be any integer. Prove that $a^{1+\ell(p-1)(q-1)} \equiv a \pmod{n}$ for all $a \in \mathbb{Z}$.

8. (4 points) Give three examples of nonisomorphic rings of order 4.

9. (10 points)

(i) Prove: If p is prime, $f(x) \in \mathbb{Z}_p[x]$ is a polynomial, and $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

(ii) Does this remain true if p is not prime? Explain.

10. (10 points)

(i) Prove that $x^2 - 3$ and $x^5 - 2$ are irreducible in $\mathbb{Q}[x]$.

(ii) Prove that $x^5 - 2$ is irreducible in $\mathbb{Q}(\sqrt{3})[x]$.

11. (8 points) Write a one or two paragraph essay explaining (without proofs) why trisecting the angle $\pi/3$ is impossible using a straightedge and a compass.